Bearing Capacity of Spatially Random Soils

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Abstract

By combining elasto-plastic finite element analysis with random field theory, a preliminary investigation has been performed into the bearing capacity of soils with spatially random shear strength. The main issue is to determine the extent to which spatial variability in soil properties affects the distribution of the computed bearing capacity. For vanishing coefficients of variation (C.O.V.) in the soil shear strength, the expected value of the bearing capacity tends to the Prandtl solution, N_c. For increasing values of C.O.V., however, the expected value of the bearing capacity falls quite steeply. The results of Monte-Carlo simulations on this nonlinear problem are presented in the form of histograms which enable a probabilistic interpretation. In particular, such plots allow the probability of overestimating the bearing capacity to be assessed.

INTRODUCTION

The paper presents results obtained using a program developed by the authors which combines nonlinear elasto-plastic finite element analysis (e.g. Smith and Griffiths 1998) with random field theory (e.g. Fenton 1990, Vanmarcke 1984). The program computes the bearing capacity of a smooth rigid strip footing (plane strain) at the surface of a weightless soil with shear strength parameters c and ϕ represented by spatially varying random fields. These two soil properties were selected to be represented as random fields since they have the greatest impact on soil bearing capacity.

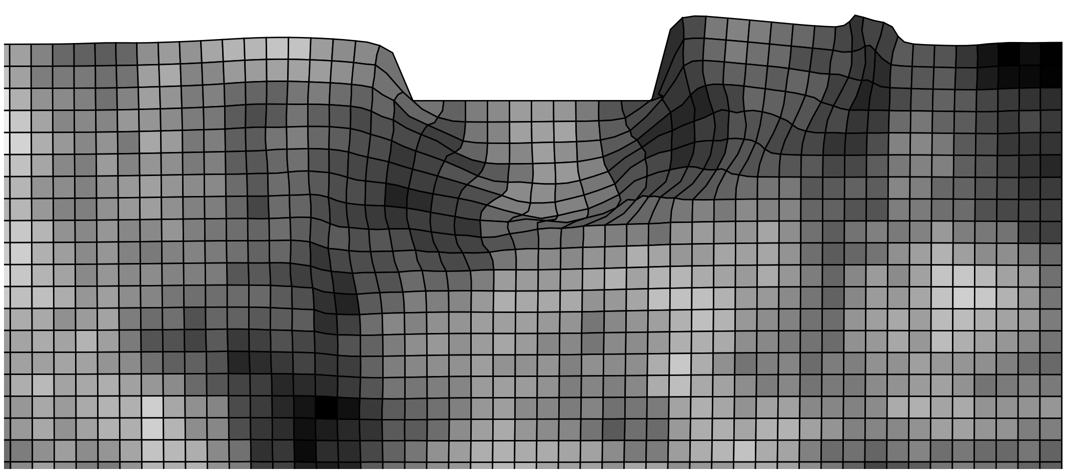

Figure 1. Typical deformed mesh at failure, where the darker regions indicate weaker soil.

Figure 1 shows a typical deformed finite element mesh resulting from a footing's bearing failure on a soil with spatially random properties. Lighter regions in the plot indicate stronger soil and darker regions indicate weaker soil. It is clear, in this case, that the weak (dark) region near the ground surface to the right of the footing has triggered a quite non-symmetric failure mechanism which is often at a lower bearing load than obtained from the traditional 'uniform' and symmetric failure analysis.

The bearing capacity analyses use an elastic-perfectly plastic stress-strain law with a Mohr-Coulomb failure criterion. Plastic stress redistribution is accomplished using a viscoplastic algorithm. The program uses 8-node quadrilateral elements and reduced integration in both the stiffness and stress redistribution parts of the algorithm. The theoretical basis of the method is described more fully in Chapter 6 of the text by Smith and Griffiths (1998). The finite element model incorporates five parameters; Young's modulus (E), Poisson's ratio (ν), dilation angle (ψ), shear strength (c), and friction angle (ϕ). The methodology allows for random distributions of all five parameters, however in the present study, E, ν and ψ are held constant (at 100000, 0.3, and 0, respectively) while c and ϕ are randomized. The finite element mesh consists of 1000 elements, 50 elements wide by 20 elements deep. Each element is a square of side length 0.1m and the strip footing occupies 10 elements, giving it a width of 1m.

Rather than deal with the actual bearing capacity, this study deals with the dimensionless bearing capacity factor, N_c, which is traditionally defined by

$$N_c = \frac{q_f}{c} \tag{1}$$

where q_f is the bearing capacity and c is the cohesion of the soil (traditionally assumed spatially constant). For a soil with spatially constant cohesion and friction angle, the theoretical bearing capacity factor, N_c, is given by Sokolovskii (1965), $N_c = (e^{\pi \tan\phi} \tan^2(45 + \phi/2) - 1)/\tan\phi$, so that, for example, if $\phi = \mu_\phi = 25$ degrees, then $N_c = 20.7$.

THE RANDOM FIELD MODEL

In this study, the soil cohesion is assumed to be lognormally distributed with mean μ_c, standard deviation σ_c, and spatial correlation length $\theta_{\ln c}$. A lognormally distributed random field is easily obtained by first simulating a normally distributed random field, $G_{\ln c}(\underset{\sim}{x})$, having zero mean, unit variance, and spatial correlation length $\theta_{\ln c}$. This 'underlying' normally distributed random field may then be transformed to the desired cohesion field using the relationship

$$c_i = \exp\{\mu_{\ln c} + \sigma_{\ln c} G_{\ln c}(\underset{\sim}{x}_i)\} \tag{2}$$

where $\underset{\sim}{x}_i$ is a vector containing the coordinates of the center of the i'th element, and c_i is the cohesion value assigned to the i'th element.

The friction angle, ϕ, is bounded both above and below, and so neither the normal nor the lognormal distributions are appropriate. In this study, a bounded distribution is selected which arises as a simple transformation of a standard normal random field, $G_\phi(\underset{\sim}{x})$. This approach again allows the generation of a normally random field followed by the transformation

$$\phi_i = \phi_{min} + \tfrac{1}{2}(\phi_{max} - \phi_{min}) \left\{ 1 + \tanh\left(\frac{sG_\phi(\underset{\sim}{x}_i)}{2\pi} \right) \right\} \tag{3}$$

where ϕ_{min} and ϕ_{max} are the minimum and maximum friction angles, respectively, and s is a scale factor which governs the friction angle variability between its two bounds. See Fenton (1990) for more details on the above transformation.

The local average random field, $G_\phi(\underset{\sim}{x})$, has zero mean and unit variance, as does $G_{\ln c}(\underset{\sim}{x})$. Conceivably, $G_\phi(\underset{\sim}{x})$ could also have its own correlation length θ_ϕ distinct from $\theta_{\ln c}$. However, it seems reasonable to assume that if the cohesion at two points are strongly correlated, then so too would be the friction angles at the same two points. Thus, θ_ϕ is taken to be equal to $\theta_{\ln c}$ in this study. Both scales will be refered to generically from now on simply as θ.

The random fields used in this study are generated using the Local Average Subdivision (LAS) method (Fenton and Vanmarcke 1990, see also Fenton 1994). A Markovian spatial correlation function is used for both fields, having the form $\rho(\tau) = \exp\{-2|\tau|/\theta\}$, where ρ is the correlation coefficient between the underlying random field values at any two point separated by a distance τ. In the two-dimensional analyses presented in this paper, the spatial correlation lengths in the vertical and horizontal directions are taken to be equal (isotropic) for simplicity.

MONTE CARLO AND PARAMETRIC STUDIES

In the parametric studies that follow, the mean cohesion (μ_c) and mean friction angle (μ_ϕ) have been held constant at 100 kN/m^2 and 25^o (with ϕ_{min} = 5^o and ϕ_{max} = 45^o), respectively, while the C.O.V. (= σ_c/μ_c) and spatial correlation length (θ) are varied systematically according to the following table

Table 1. Random field statistics used in the parametric study.

θ	=	0.5	1.0	2.0	4.0	8.0	50.
C.O.V.	=	0.1	0.2	0.5	1.0	2.0	5.0

In addition, it is assumed that when the variability in the cohesion is large, the variability in the friction angle will also be large. Under this reasoning, the scale factor, s, used in Eq. (3) is set to $s = 2\sigma_c/\mu_c$ = 2(C.O.V.). This choice is arbitrary, but results in the friction angle varying quite narrowly (when C.O.V. = 0.1 and $s = 0.2$) to very widely (when C.O.V. = 5.0 and s = 10) between its lower and upper bounds, 5^o and 45^o. Also, in general, cohesion and friction angle can be correlated to some extent. However, in this preliminary study, the two fields are assumed to be independent – ongoing simulations are looking into the issue of correlation between the fields.

For each set of assumed statistical properties given by Table 1, Monte-Carlo simulations have been performed. These involve 100 repetitions or "realizations" of the soil property random fields and the subsequent finite element analysis of bearing capacity. Each realization, therefore, has a different value of the bearing capacity and, after normalization by the *mean* cohesion, a different value of the bearing capacity factor,

$$N_{c_i} = \frac{q_{f_i}}{\mu_c}, \quad i = 1, 2, \dots, 100, \quad \Longrightarrow \quad \bar{N_c} = \frac{1}{100}\sum_{i=1}^{100} N_{c_i} \tag{4}$$

SIMULATION RESULTS

Figure 2(a) shows how the sample mean bearing capacity factor, taken as the average of the N_{c_i} computed over all soil realizations, and refered to as $\bar{N_c}$, varies with the correlation length and soil variability. For small soil variability, $\bar{N_c}$ tends towards the deterministic value of 20.7, which is found when the soil takes on its mean properties everywhere. For increasing soil variability, the mean bearing capacity factor becomes quite significantly reduced from the ideal case. What this implies from a design standpoint is that the bearing capacity of a heterogeneous soil will, on average, be *less* than the Prandtl solution which would be predicted assuming the soil has strength given by mean values. The greatest reduction from the Prandtl solution is observed for low θ values. This can be explained by the fact that as the correlation length decreases, the possibility of a lower strength highly non-symmetric bearing failure increases because of the more rapidly varying (in space) soil properties.

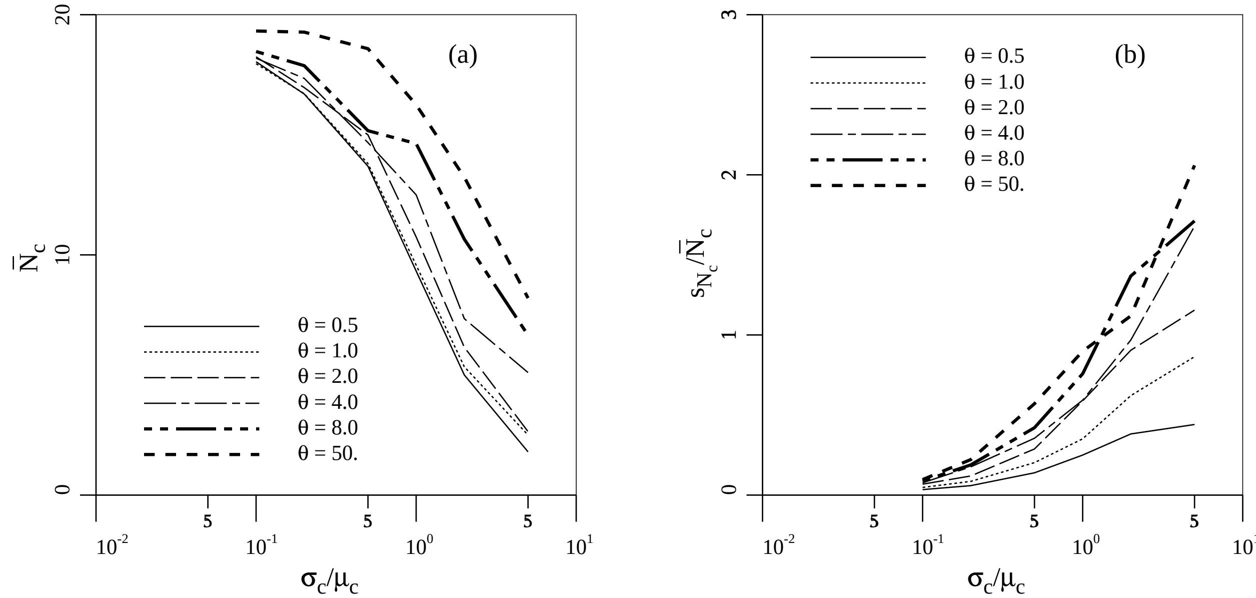

Figure 2. a) Sample mean bearing capacity factor, $\bar{N}_c$, and b) sample coefficient of variation of N_c.

PROBABILISTIC INTERPRETATION

Following Monte-Carlo simulations for each parametric combination of input parameters (θ and C.O.V.), the suite of computed bearing capacity factor values from Eq. (4) was plotted in the form of a histogram, and a "best-fit" lognormal distribution superimposed. An example of such a plot is shown in Figure 3 for the case where θ = 2 and C.O.V.= 1.

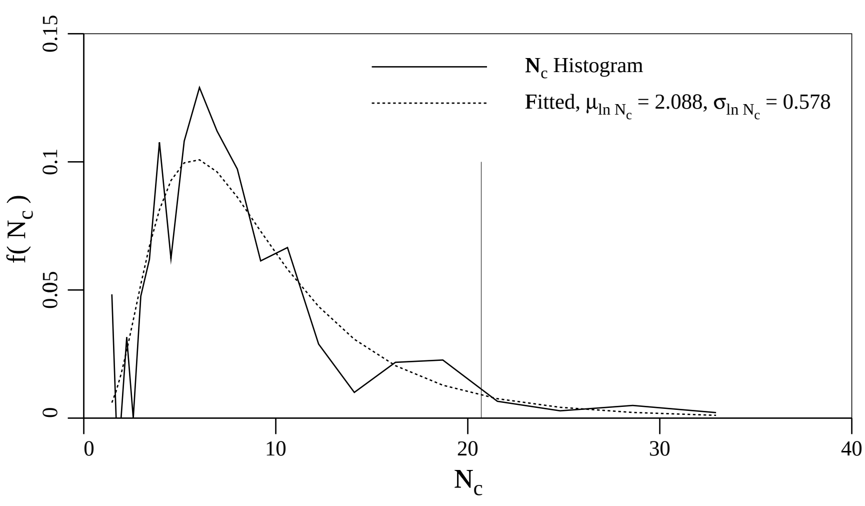

Figure 3. Typical normalized histogram of N_c values with superimposed fitted lognormal distribution.

Since the lognormal fit has been normalized to enclose an area of unity, areas under the curve can be directly related to probabilities. From a practical viewpoint it would be of interest to estimate the probability of "design failure", defined here as occurring when the computed bearing capacity is less than the Prandtl value based on the mean soil properties, i.e. we have design failure if $N_c < 20.7$, where N_c is computed from Eq. (1).

Assuming that N_c does follow a lognormal distribution, as is roughly indicated by Figure 3, the "design failure" probability can be computed as

$$\mathrm{P}\left[N_c < 20.7\right] = \Phi\left(\frac{\ln 20.7 - \mu_{\ln N_c}}{\sigma_{\ln N_c}}\right) \tag{5}$$

where Φ is the cumulative normal function. For the particular case shown in Figure 3, the fitted lognormal distribution has parameters $\mu_{\ln N_c} = 2.088$ and $\sigma_{\ln N_c} = 0.578$. Eq. (5) gives $\mathrm{P}\left[N_c < 20.7\right] = 0.95$, indicating an 95% probability that the actual bearing capacity will be less than the Prandtl value.

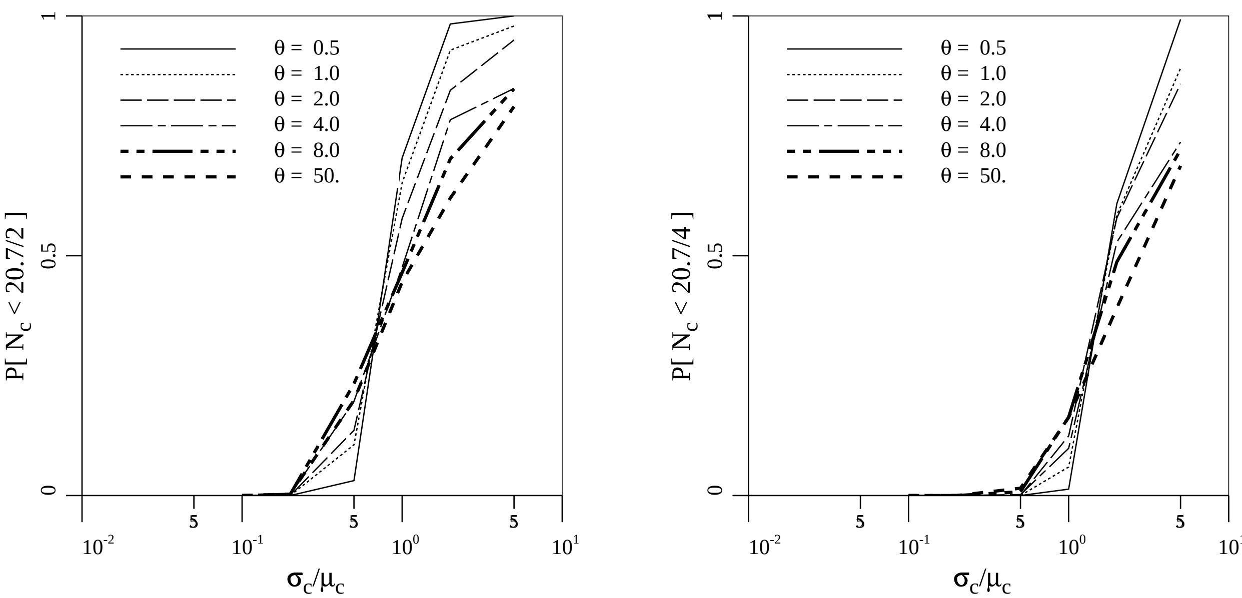

Figure 4. Effect of factor of safety on the probability of design failure, $\mathrm{P}\left[N_c < 20.7/F\right]$**, for a)** $F = 2$**, and b)** $F = 4$**.**

Figure 2(a) indicates that the expected bearing capacity of a strip footing on a soil with spatially variable cohesion and friction angle will *always* be lower than the Prandtl value based on the mean soil. However, the design capacity is generally based on the Prandtl solution reduced by a "Factor of Safety", F. The probability of design failure, in the form of $\mathrm{P}\left[N_c < 20.7/F\right]$, is considerably reduced, giving a more reassuring result from a design viewpoint. Figure 4 illustrates the probability of design failure for two different factors of safety. For example, from Figure 4(a), in which $F = 2$, the probability of design failure for a soil with $\theta = 4$ and C.O.V.= 0.5, is about 14%. This probability is reduced to about 0.2% for the same soil, by increasing the factor to $F = 4$, as shown in Figure 4(b).

These plots indicate that quite high factors of safety are required to reduce the probability of "design failure" to negligible levels. The most important factor affecting the probability of design failure appears to be the soil variability (which includes the cohesion and friction angle variabilities). The correlation length, under the assumed model, has only a secondary affect on the magnitude of the probability of design failure.

These results may help explain in a probabilistic context, why Factors of Safety used in bearing capacity calculations are typically higher than those used in other limit state calculations in geotechnical engineering, e.g. slope stability, earth pressures.

CONCLUDING REMARKS

The paper has shown that soil strength variability can significantly reduce the mean bearing capacity of a strip footing on a c-phi soil.

It should be emphasized that this study is still preliminary. Only 100 realizations were considered due to time constraints (the non-linear bearing problem is *extremely* computer intensive, taking about 5 to 6 hours to run 100 realizations of a single parameter set on a 500 MHz P-III computer), nor were all aspects of the problem thoroughly investigated – the research is ongoing. However, from this initial study, the following more specific observations can be made:

1) As the variance of the soil strength increases, the mean bearing capacity decreases. The magnitude of decrease of the mean bearing capacity is greatest for small correlation lengths.

2) As the C.O.V. of the soil strength increases from zero, the C.O.V. of the bearing capacity also increases. Increasing the spatial correlation length consistently increases the C.O.V. of the bearing capacity.

3) Results have been presented in a probabilistic context to determine the probabitity of "design failure", defined as the probability that the actual bearing capacity would be lower than a deterministic prediction of factored bearing capacity using Prandtl's formula based on the mean strength of the soil.

4) By investigating the role of a Factor of Safety applied to the Prandtl solution, it was observed that a value of F = 4 and greater may be required to reduce the probability of "design failure" for soils to a negligible amount.

5) The influence of the correlation length on the probabilistic interpretation of the bearing capacity problem was seen to be not greatly significant, within the range of lengths considered. The major factor influencing the probability of a "design failure" is the soil C.O.V.

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