Seeking out failure: The Random Finite Element Method (RFEM) in probabilistic geotechnical analysis

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Abstract
The paper describes probabilistic analyses in geotechnical engineering which use the Random Finite Element Method (RFEM). The method rigorously includes the influence of spatially variable soil properties on design outcomes relating to familiar geotechnical analyses such as bearing capacity, slope stability and limiting earth pressure. Two important observations from these studies are highlighted: Firstly, by making no *a priori* assumptions about the shape or location of the critical failure mechanism, the RFEM allows the mechanism to "seek out" the critical path through the soil. Secondly, the analyses indicate that there is a critical spatial correlation length that leads to a minimum strength of the soil mass.

Introduction

The Random Finite Element Method (RFEM) represents a rigorous probabilistic approach for modeling the influence of spatially variable soil properties on design outcomes in geotechnical engineering. The method allows statements to be made about the probability of events occurring such as the probability of bearing capacity failure, or the probability of a slope or mine pillar collapse. The RFEM has been described in detail elsewhere, but essentially involves a marriage of Random Field Theory and finite element analysis. In the RFEM developed by the authors, the random field is generated using the Local Average Subdivision method (Fenton and Vanmarcke 1990) which is then mapped onto finite element meshes. Monte-Carlo simulations then follow until the probabilities of design outcome(s) of interest become statistically accurate. The authors have applied this methodology to a wide range of classical geotechnical problems ranging from linear analyses of steady seepage and settlement to nonlinear analyses of soil failure. Both 2-D and 3-D RFEM analyses have been reported and the interested reader is referred to `www.engmath.dal.ca/rfem/rfem.html` for a summary of these publications. The RFEM software used in these studies make use of the open-source finite element codes of Smith and Griffiths (2004) which are available for full download at `www.mines.edu/fs_home/vgriffit/4th_ed/Software`.

In this paper we will concentrate on the RFEM applications involving failure of soils masses such as bearing capacity, slope stability and limiting earth pressure. Two particular issues will be highlighted. Firstly the ability of the RFEM to allow the failure mechanism to "seek out" the most critical path through the soil mass, and secondly the observation that there exists a critical spatial correlation length that leads to a minimum factor of safety.

The first observation is actually a consequence of upper-bound theorems of structural plasticity (e.g. Horne 1971), namely that, " An external load computed on the basis of an assumed mechanism, in which the forces are in equilibrium is always greater than or equal to the true collapse load." This implies that if a failure mechanism is assumed *a priori* in a collapse calculation, it will *always* overestimate the collapse load unless it happens to be the correct one. In classical problems of geotechnical analysis involving homogeneous soils, the critical failure mechanism is usually well established. For example the limiting passive load against a smooth vertical wall translated into a bed of homogeneous frictional soil is reached when a planar failure mechanism inclined at $45° - \phi'/2$ to the horizontal is developed. In slope stability analysis of homogeneous soils with $\phi = 0$, charts have been developed for identifying the location of the circular mechanism depending on the geometry (e.g. Duncan and Wright 2005).

In non-homogeneous soils, the situation is quite different. From the upper-bound theorem mentioned above, it will be unconservative to assume a failure mechanism as if the soil was homogeneous, even if the soil properties are correctly "averaged " to account for variability. El Ramly *et al.* (2002) reported probabilistic slope stability analyses of a layered slope using both simplified and more sophisticated approaches. The latter approaches included local averaging, which led to variance reductions dependent on spatial correlation over certain arc lengths of the circular failure surface assumed by Bishop's method (1955). Quite apart from probabilistic considerations, it was subsequently pointed out by Duncan *et al* (2003), that the assumption of a circular failure surface in the layered example under consideration, led to a quite significant overestimation of the factor of safety. Using Spencer's method (1967), the factor of safety assuming a "wedge shaped and curved noncircular surface" fell to 1.17. The probabilistic implications of using the "wrong" failure surface were shown to be even more significant in terms of the slope reliability. Not only that, but the assumption of a fixed failure surface is unconservative because the failure mechanism is inevitably forced to pass through high strength regions in the soil.

Rather than debate which decades-old classical method is more appropriate for analysis of non-homogeneous slopes, we want to highlight the advantages of the elasto-plastic finite element (FE) slope stability approach for both conventional and probabilistic slope stability studies. The method has been known in the academic community for many years (Smith and Hobbs 1974, Zienkiewicz *et al.* 1975) and is a powerful alternative to classical slope stability methods. Source software using this method has been published in Smith and Griffiths (2004) and described in detail elsewhere (e.g. Griffiths and Lane 1999). A key advantage of the FE method as applied to any limit analysis in geotechnics (e.g. slope stability, bearing capacity, earth pressures) is that the analysis allows the soil mass to "fail where it wants to fail". In other words, it is entirely unnecessary to anticipate the shape or location of the critical failure surface *a priori* since this information comes out of the analysis automatically. While the benefits of the FE approach are clear for any non-homogeneous problem, they are even more important when dealing with highly variable conditions such as those usually implied when the soil properties are described statistically. In this case we have zones of relatively weak soil interspersed with zones of relatively strong soil, where the size of the zones is a function of the spatial correlation length. In such a variable soil, failure will not occur simultaneously at all points along a simple circular path as assumed, for example, in Bishop's method, but could follow quite convoluted paths as it "seeks out" the failure path through the soil requiring the minimum amount of energy. In an actual soil mass, failure will be progressive and attracted to the weaker zones. Triggering of failure in the weak zones will then throw more stress onto neighboring regions which will fail in their turn. Eventually the failing zones of soil join to form a mechanism leading to overall failure. The RFEM approach is able to model this progressive failure effect and "seek out" the weakest path through the soil. The effects of spatial correlation are fully modeled by using a refined mesh, and local averaging accounted for at the finite element level. Unlike the classical approaches, the user does not have to assume any averaging domain *a priori* since the strength averaging occurs naturally as a result of the failure mechanism "seeking out" the critical failure path through the soil.

Although the strength of a soil may be defined in terms of a mean, standard deviation and correlation length, from a probabilistic point of view, the locations of the weak and strong zones referred to above are unknown. If the weak zones happen to occur near the toe or under the middle of the slope, the probability of failure will be relatively high, whereas if strong zones occurs at those locations, the probability will be relatively low. Both analyses have the same underlying input statistics, but the spatial distribution of the properties can have a profound effect on stability. For this reason the RFEM involves Monte-Carlo simulations in which the stability analysis is repeated until the probabilities relating to output quantities of interest become statistically accurate. In the case of a slope stability analysis we would typically be interested in the probability of failure, which would be estimated by dividing the number of realizations in which the slope failed by the total number of realizations. For a bearing capacity analysis, the number of realizations in which the bearing capacity dropped below some design threshold divided by the total number of realizations might be the probability estimate of interest. It should be noted that the RFEM is quite computationally intensive when modeling elasto-plastic (Mohr-Coulomb) materials, since each Monte-Carlo simulation involves a full nonlinear analysis involving iterative redistribution of stresses.

Slope stability analysis

Figure 1 shows a RFEM test problem involving a slope with $\phi = 0$ and constant unit weight (γ), as considered by Griffiths and Fenton (2004). The undrained shear strength (c_u) is characterized by a mean (μ_{c_u}), a standard deviation (σ_{c_u}) and a spatial correlation length $(\theta_{\ln c_u})$. The analyses were performed in non-dimensional form by relating all results to a strength parameter $C = c_u / \gamma H$. The spatial correlation length was also non-dimensionalized by relating it to the height of the slope where $\Theta_C = \theta_{\ln c_u} / H$. Some typical results of failed slopes are shown in Figure 2.

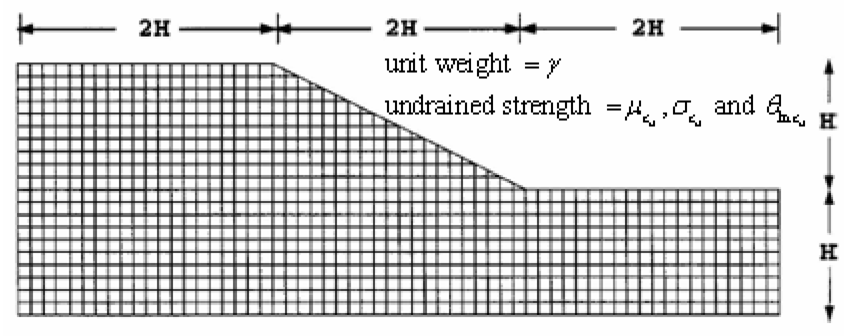

Figure 1. Test problem involving a slope with $\phi = 0$

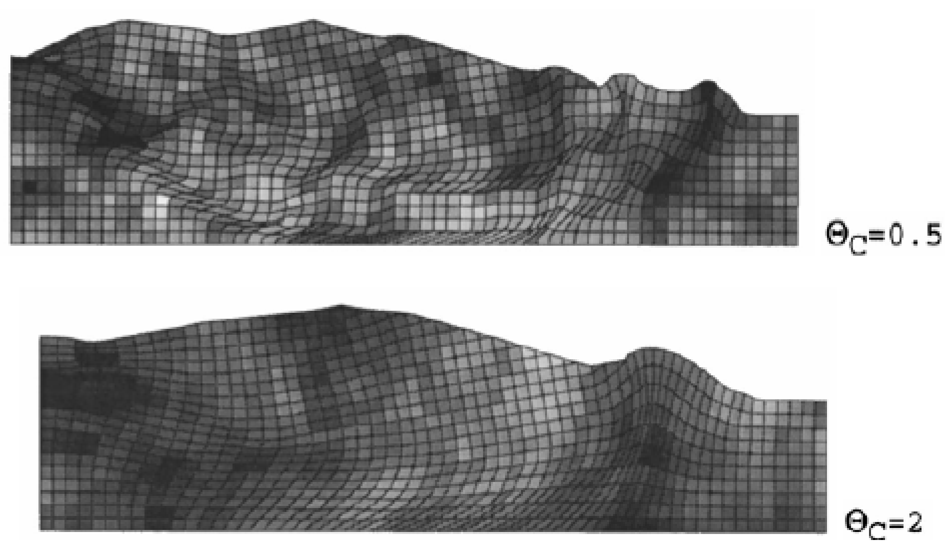

Figure 2. Typical random fields and deformed meshes at failure in a slope stability analysis for two different spatial correlation lengths (dark regions indicate weak soil)

Although both these analyses shown in Figure 2 had the same mean and variance, the different spatial correlation lengths have led to quite different failure characteristics. In the case where $\Theta_C = 0.5$**,** the RFEM indicates a quite complex failure mode with multiple competing mechanisms. Such a failure mode would defy analysis by conventional slope stability analysis tools. In the case where $\Theta_C = 2.0$**,** a much smoother failure mechanism was observed more like the type of mechanism that would be observed in a homogeneous soil. The role of Θ_C has been discussed in detail elsewhere, where it has been observed that as $\Theta_C \to \infty$ and as $\Theta_C \to 0$ the mechanisms tend to become smoother, whereas at intermediate values of Θ_C the mechanisms reach a maximum level of tortuosity. In the slopes considered so far by RFEM, this maximum was observed to lie in the range $0.5 < \Theta_C < 1$.

Bearing capacity

Figure 3 shows a typical failure mechanism, this time in the form of nodal displacement vectors, as observed in an RFEM analysis of bearing capacity (Fenton and Griffiths 2003). As with Figure 2, this represents just a typical observation from a suite of Monte-Carlo simulations, however it demonstrates the unpredictability of the failure mechanism as it seeks out the weakest path through the soils below the footing. The mechanism is also clearly non-

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symmetric, which is more representative of actual bearing capacity failures than the perfect symmetry of classical theories.

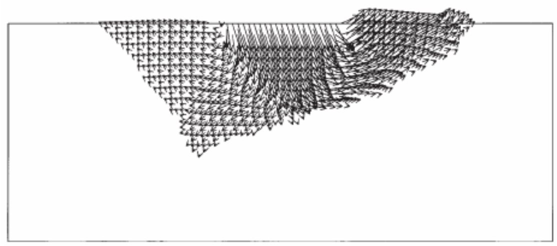

Figure 3. Typical displacement vectors at failure in a bearing capacity analysis by RFEM

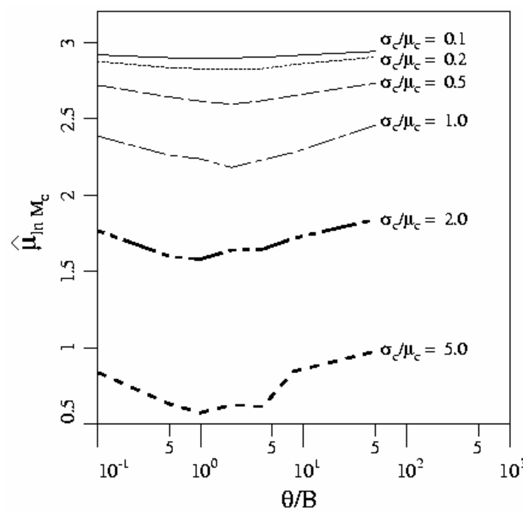

Figure 4. Sample mean of log bearing capacity factor, $\ln M_C$, versus normalized correlation length.

Figure 4 indicates the influence of spatial correlation length (θ/B, non dimensionalized with respect to the footing width B) on the bearing capacity factor M_c (representing the estimated value of the familiar bearing capacity factor N_c) for a range of input coefficients of variation of input shear strength (σ_c/μ_c). The variation of $\mu_{\ln M_c}$ with respect to θ is seen to reach a minimum when θ is of the same order as the footing width B. This is a consequence of the "seeking out" effect of the RFEM. It is hypothesized that $\theta \approx B$ leads to the greatest reduction in $\mu_{\ln M_c}$ because it allows enough spatial variability for a failure surface to form that deviates somewhat from the theoretically based log-spiral result but whose path is not too long (as occurs when θ is too small), yet has a significantly lower strength average than when $\theta \to \infty$. Perhaps more importantly, since little is generally known about the correlation length at a site, the results of this study indicate that there is a worst-case correlation length of $\theta \approx B$. Using this value in the absence of improved information, allows conservative estimates of the probability of bearing failure. The bearing capacity design methodology developed by Fenton and Griffiths (2003) is based on this conservative case.

Passive earth Pressure

The final application of the RFEM to be described in this paper involves passive earth pressure (Griffiths *et al.* 2006). The practical implications for design lie in estimating the probability that the passive resistance of an anchorage system would fall below some design threshold. The design value might be based on a Rankine calculation using the mean strength and reduced by a factor of safety. Figure 5 shows a typical RFEM mesh at failure following translation of a smooth vertical wall into a bed of soil with a horizontal backfill. The figure clearly displays a "Rankine-type" mechanism heading towards the ground surface from the base of the wall, but there are also secondary mechanisms heading back towards the top of the wall "seeking out" the weaker soils in that region.

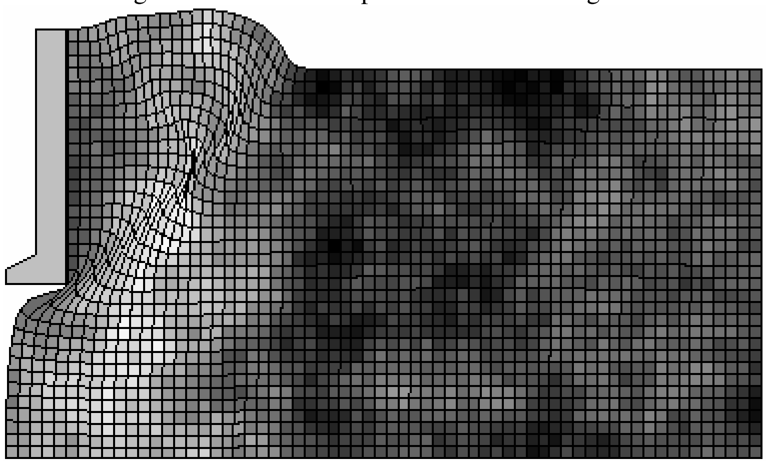

Figure 5. Typical random field and deformed mesh at failure in a passive earth pressure analysis (light regions indicate weak soil)

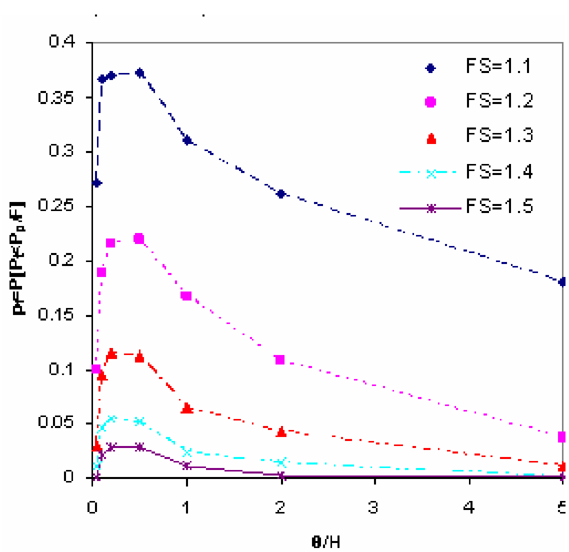

Figure 6. Estimated probability that true passive resistance load is less than a factored design value.

As in the bearing capacity analyses, a critical spatial correlation length of the order $0.5 < \theta < 1$ (where H is the wall height) was observed, which led to the highest probability of passive failure. This effect is shown clearly in Figure 6 where the probability of the passive resistance falling below a factored Rankine value is plotted against a range of θ/H values. Greater detail on the passive earth pressure studies by RFEM can be found in Griffiths *et al.* (2006) where the influence of taking a "virtual sample" from the random field was also investigated.

Concluding Remarks

The paper has demonstrated features of the Random Finite Element (RFEM) as applied to probabilistic analysis of some familiar geotechnical problems involving failure of soil masses. Two important benefits of the RFEM over more traditional probabilistic approaches were highlighted.

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- By making no assumption *a priori* about the shape or location of the failure surface the RFEM is able to account for property "averaging" in a natural way by allowing the mechanism to "seek out" the critical path through the soil mass. This is in contrast to methods that anticipate the failure surface in advance, often based on homogeneous soil mechanics assumptions, and then perform local averaging over that surface. Based on classical upper-bound plasticity theorems, these other approaches will likely overestimate the strength of the soil mass and therefore underestimate the probability of failure.

- A consequence of using the RFEM with reasonably refined meshes (where the element size is significantly smaller than the spatial correlation length θ), is that the "seeking out" effect indicates a worst-case correlation length (θ_{crit}) that leads to a maximum probability of failure. It is hypothesized that at this critical length, the RFEM approach allows the mechanism enough freedom to deviate from the classical path and seek out weaker soils. If θ becomes too small, the mechanism chooses to cut through some stronger soils than seek out weaker soils along a very tortuous path. Alternatively, as θ becomes large, the soil is becoming more homogeneous, hence the mechanisms are also tending to follow the paths dictated by classical (homogeneous) soil mechanics.

Acknowledgements

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