THE INFLUENCE OF STRENGTH VARIABILITY IN THE ANALYSIS OF SLOPE FAILURE RISK

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ABSTRACT: Probabilistic analysis of failure problems in geomechanics is often directed towards assessing the mean and variance of design quantities (e.g. Factor of Safety, bearing capacity, limiting earth pressure) as a function of the mean and variance of input quantities (e.g. shear strength parameters). When spatial correlation length is also included as an input parameter, an additional complexity is introduced in that this parameter directly impacts the locally averaged shear strength along a failure surface. A key advantage of the Random Finite Element Method (RFEM) over conventional methods is that no *a priori* assumptions are made about the shape or location of the critical failure mechanism. The RFEM enables the mechanism to "seek out" the critical route leading to the minimum factor of safety. By forcing the mechanism to be circular (say), traditional approaches are inevitably "upper bound" and can lead to unconservative conclusions regarding slope failure risk.

INTRODUCTION

The majority of probabilistic slope stability analyses continue to use classical slope stability analysis techniques (e.g., Bishop 1955). The deficiency of traditional slope stability approaches is that the shape of the failure surface is often fixed by the method and the failure mechanism is not allowed to "seek out" the most critical path through the soil. Second, spatial correlation and local averaging of statistical geotechnical properties has typically been omitted from many probabilistic slope stability analyses.

More recently, spatial correlation has been included in probabilistic slope stability analysis by modeling the soil parameters as 1D random fields (e.g., El-Ramly et al. 2002). In this approach, the critical failure surface over which the 1D random fields are assumed is typically determined using a traditional slope stability method using locally averaged soil properties. The influence of the local averaging depends on the spatial correlation length and the correlation function as will be discussed later. The results of this analysis can be sensitive to the choice of failure surface (e.g., circular)

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and may lead to an overestimation of the factor of safety and hence underestimate the probability of failure (e.g. Duncan et al. 2003).

To avoid having to select the appropriate shape of the failure surface, the elasto-plastic finite element (FE) slope stability approach can be applied. This approach, known in the academic community for years (Smith and Hobbs 1974, Zienkiewicz et al. 1975), has the key advantage that the analysis allows the soil mass to "fail where it wants to fail". It is not necessary to anticipate the shape or location of the critical failure surface *a priori* since this information comes out of the analysis automatically. To create an even more powerful tool, the FE approach has been joined with random field theory to provide a rigorous probabilistic approach for modeling the influence of statistically described soil properties on design outcomes in geotechnical engineering. The marriage of random field theory and FE analysis, called the "random finite-element method" (RFEM), takes full account of local averaging, variance reduction, and spatial correlation without the disadvantage of anticipating a potential failure surface (Griffiths and Fenton 1993 and see www.engmath.dal.ca/rfem/rfem.html for a full list of RFEM publications).

To demonstrate the benefits of RFEM the probabilistic stability characteristics of a cohesive slope will be investigated using both a simplified slope stability analysis and RFEM.

The slope under consideration is shown in Figure 1, and consists of undrained clay, with shear strength parameters ϕ_u=0 and c_u. This is the same test slope considered by Griffiths and Fenton 2004. In this study, the slope inclination and dimensions, give by β, H, and D, and the saturated unit weight of the soil, γ_{sat}, are held constant, while the undrained shear strength c_u is assumed to be a random variable. The undrained shear strength is conveniently expressed in the dimensionless form C, where $C=c_u/(\gamma_{sat}H)$.

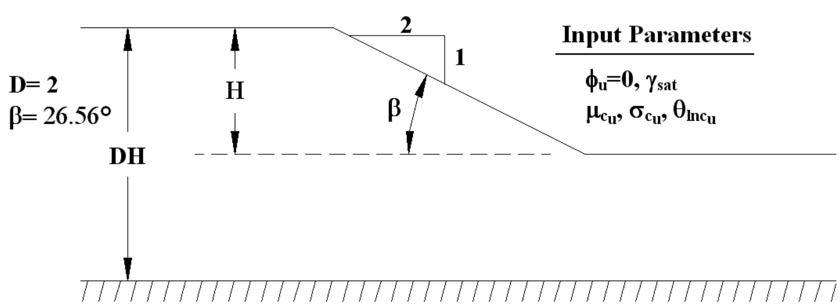

Fig. 1. Cohesive slope

PROBABILISTIC DESCRIPTION OF SHEAR STRENGTH

The shear strength, C, is assumed to be characterized statistically by a log-normal distribution defined by a mean, μ_C, and a standard deviation, σ_C. The mean and standard deviation can be expressed in terms of the dimensionless coefficient of variation, defined as

$$V_C = \frac{\sigma_C}{\mu_C} \quad \textbf{(1)}$$

The third parameter considered is the spatial correlation length, $\theta_{\ln C}$. The spatial correlation length ($\theta_{\ln C}$) describes the distance over which the spatially random values will tend to be significantly correlated. Thus, a large value of $\theta_{\ln C}$ will imply a smoothly varying field, while a small value will imply a rough field. In this study, the spatial correlation length has been normalized by the slope height and will be expressed in the form,

$$\Theta_C = \frac{\theta_{\ln C}}{H} \quad \textbf{(2)}$$

It has been suggested (see, e.g., Lee et al. 1983; Phoon & Kulhawy 1999) that typical V_C values for undrained shear strength are in the range of 0.1-0.5. The spatial correlation length is less well documented and may exhibit anisotropy. Although the capability of modeling an anisotropic soil is available, the spatial correlation considered in this study will be assumed to be isotropic.

DETERMINISTIC STUDY

An initial deterministic study assuming a homogeneous soil has been performed to put the probabilistic analyses into context. For the simple slope shown in Figure 1, the factor of safety was obtained using simple equilibrium methods to give the data in Table 1. From Figure 2, a linear relationship exists between the factor of safety, FS, and the shear strength, C, which leads to Equation (3).

$$FS = 5.88C \quad \textbf{(3)}$$

Table 1. Factors of Safety Assuming Homogenous Soil

C	Factor of Safety (FS)
0.15	0.88
0.17	1.00
0.20	1.18
0.25	1.47
0.30	1.77

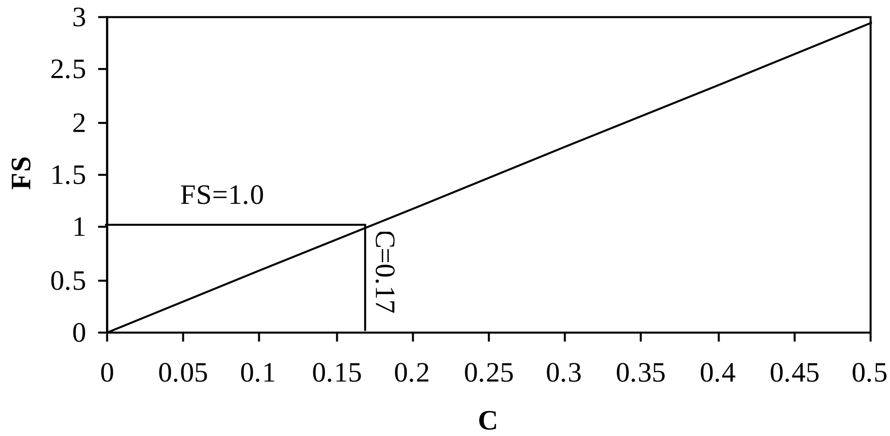

Fig. 2. Linear relationship between factor of safety and C for a cohesive slope with slope angle β=26.57º and a depth ratio of D=2

SIMPLE APPROACH

The analysis presented here investigates the influence of giving the shear strength C a log-normal probability density function based on a mean, a standard deviation and spatial correlation length. A locally averaged value of C based on 1D averaging over an arc length as shown in Figure 3 will be used in Equation 3.

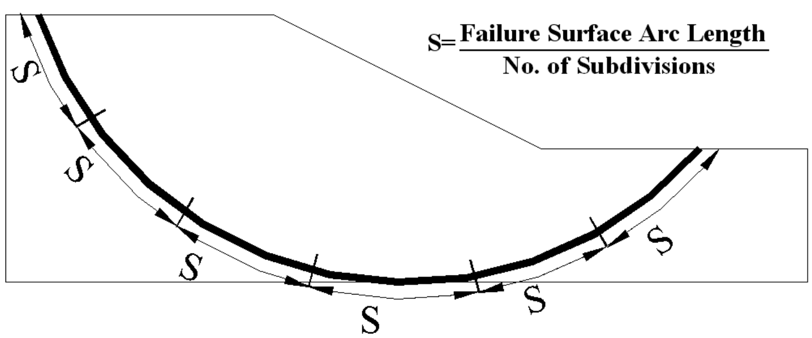

Fig. 3. Assumed failure surface with example failure surface divisions

To account for local averaging and spatial correlation, the failure circle will be divided into equal arc lengths and the soil properties averaged along each arc. The variance reduction factor due to local averaging, γ, is defined as

$$\gamma = \left(\frac{\sigma^2_{\ln C_A}}{\sigma^2_{\ln C}} \right) \qquad \textbf{(4)}$$

where the subscript A refers to the locally averaged statistic over an arc length. The variance reduction factor is a function of the slice length and the correlation function.

The correlation function used is an exponentially decaying (Markovian) correlation function of the following form

$$\rho = e^{-\left(2S/\theta_{\ln c}\right)} \quad \textbf{(5)}$$

where *S* is the arc length for local averaging.

For a line of length $S=\alpha\theta_{\ln C}$ it can be shown (Vanmarcke 1984) that the variance reduction factor is given by

$$\gamma = \frac{2}{(\alpha\theta)^2}\int_0^{\alpha\theta} e^{\left(-2\frac{x}{\theta}\right)}(\alpha\theta - x)\,dx \quad \textbf{(6)}$$

Performing the integration leads to the variance reduction values plotted in Figure 4. Figure 4 illustrates that arc length impacts variance reduction. For line lengths that are small with respect to correlation length, there is little reduction in variance, whereas line lengths that are large relative to correlation length can result in significant reduction.

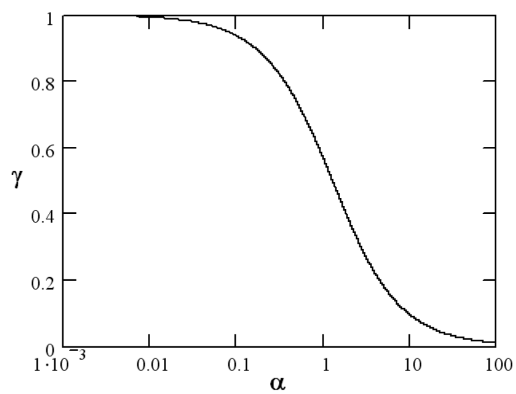

Fig. 4. Variance reduction function over a line of length $\alpha\theta_{\ln C}$ with a Markov correlation function

The underlying statistics including local averaging are given by

$$\sigma_{\ln C_A} = \sigma_{\ln C}\sqrt{\gamma} \quad \textbf{(7)}$$

$$\mu_{\ln C_A} = \mu_{\ln C} \quad \textbf{(8)}$$

Which leads to the following statistics including local averaging.

$$\mu_{C_A} = e^{\left(\mu_{\ln C_A} + \frac{1}{2}\sigma^2_{\ln C_A}\right)} \quad \textbf{(9)}$$

$$\sigma_{C_A} = \mu_{C\_L}\sqrt{e^{\left(\sigma^2_{\ln C_A}\right)} - 1} \quad \textbf{(10)}$$

For the problem considered, the surface is divided into equal slices. The locally averaged statistics will therefore be the same for each slice and Equation (3) can be used with the locally averaged statistics to determine the probability of failure of the slope, which is defined as the probability that the factor of safety will be less than 1.0.

The influence of the number of subdivisions on the locally averaged statistics is illustrated in Figure 5. A large number of slices indicate that the line length is small with respect to the spatial correlation length and the locally averaged statistics closely resemble the input point statistics. Conversely, for a fewer number of subdivisions, the line lengths are large with respect the spatial correlation length and the locally averaged statistics are reduced with respect to the input statistics.

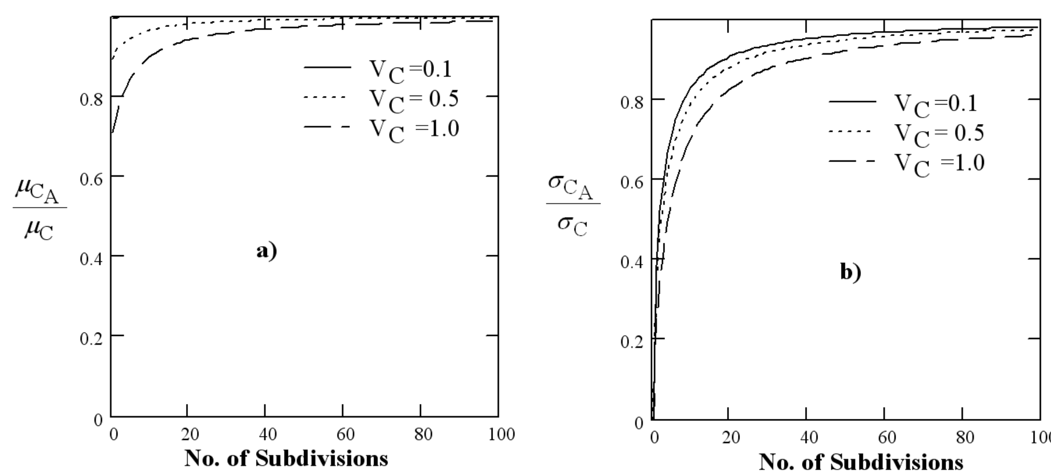

Fig. 5. Influence of number of subdivisions on local averaging: influence on the (a) mean and (b) standard deviation

Using properties derived from the above local averaging theory in Equation (3) and performing a first order second moment (FOSM) method to determine the probability that the factor of safety will be less than 1.0, with the point mean fixed at μ_C=0.25 leads to Figures 6 and 7. Figure 7 clearly shows the influence of the length over which the local averaging is occurring. All curves cross over at a critical value of V_C = 1.08, which is the value of V_C which results in the $Median_C$ = 0.17 (see Table 1). This result is similar to the study presented by Griffiths and Fenton (2004) where a critical value of V_C was determined by performing variance reduction over a square element.

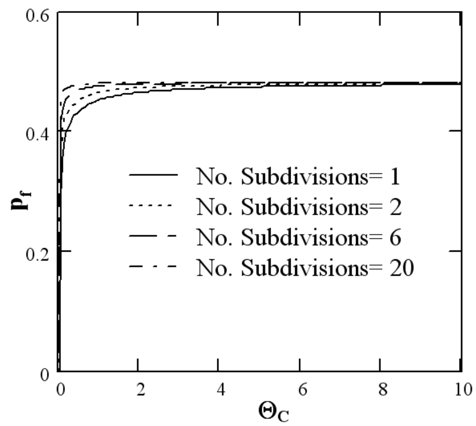

Fig. 6. The probability of failure versus spatial correlation length for different numbers of subdivisions; the mean and coefficient of variation are fixed at μ_C=0.25 and V_C=1.0, respectively

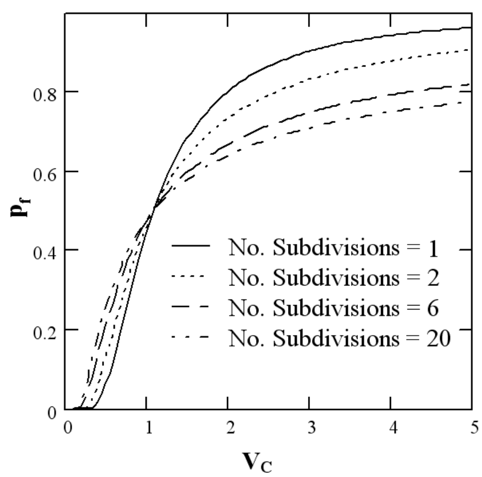

Fig. 7. The probability of failure versus coefficient of variation for different numbers of subdivisions; the mean and special correlation length are fixed at μ_C=0.25 and Θ_C=1.0, respectively

RANDOM FINITE-ELEMENT METHOD

The RFEM enables soil property variability and spatial correlation to be accounted for in a rigorous and general way. The method involves the generation and mapping of a random field of shear strength values onto a refined finite element mesh. Full account is taken of local averaging and variance reduction at the element level, and a spatial correlation function (Equation 5) is incorporated. An elasto-plastic finite element analysis is then performed using a Mohr-Coulomb failure criterion (see e.g. Griffiths & Fenton, 2001). For a given set of input shear strength parameters (μ_C, σ_C, $\theta_{\ln C}$), 1000 simulations are performed for each set of input parameters and the probability of failure is defined as the proportion of 1,000 Monte Carlo slope stability analyses that failed.

Figures 8 (a and b) show typical meshes and the influence of spatial correlation length on the shear strength values populating the mesh. Dark and light regions represent "weak" and "strong" soils, respectively.

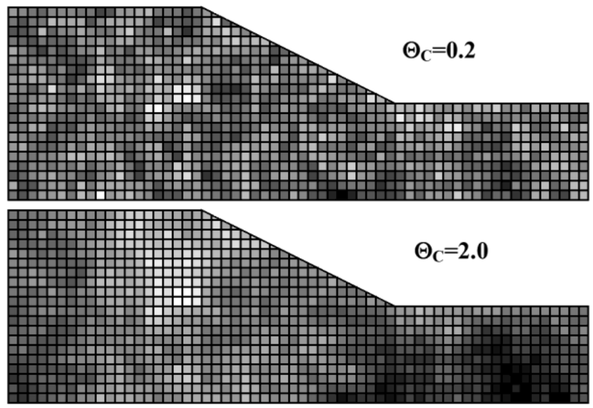

Fig. 8. Influence of spatial correlation in RFEM

In the RFEM approach, the failure mechanism is free to "seek out" the weakest path through the soil. This is illustrated in Figures 9 (a and b) where the shear strain invariant has been contoured at failure for two Monte Carlo simulations. Dark areas represent a high value of the shear strain invariant and light areas represent low values. To emphasize the "seeking out" effect, the critical failure surface given by Bishop's method is shown. Figure 9a clearly shows a failure mechanism that is quite different from that determined by classical limit equilibrium methods.

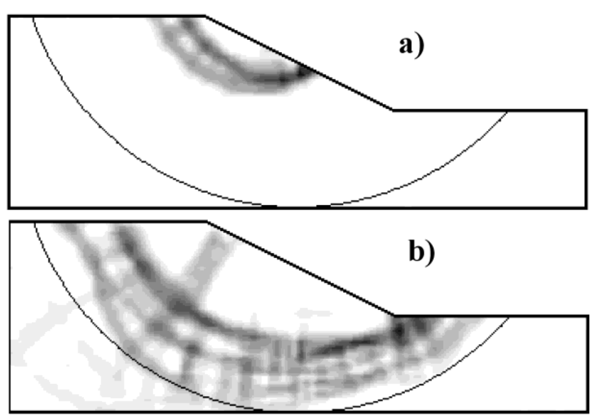

Fig. 9. Strain invariant contour for μ_C=0.25, V_C=1.0 and Θ_C=1.0

The results from the RFEM are shown in Figure 10.

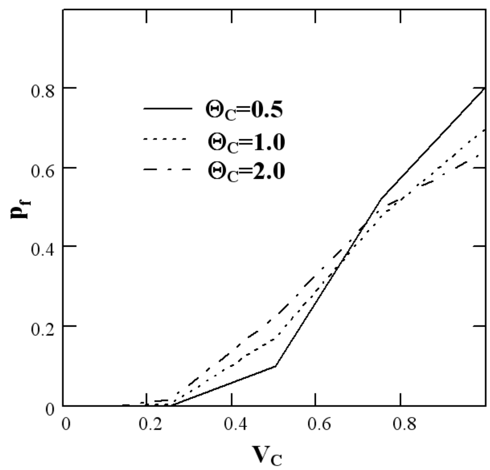

Fig. 10. The probability of failure versus coefficient of variation based on RFEM; the mean is fixed at μ_C=0.25

To better illustrate the differences in the predicted probabilities of failure by the two techniques, Figures 11 (a and b) show the results of both the simple approach and RFEM. Figure 11 shows that by taking too few subdivisions, the simple approach can lead to an unconservative prediction of the probability of failure. This effect is particularly noticeable for higher values of say $V_C > 0.5$. This can be expected as the assumed failure surface does not always represent the failure mechanism of a spatially random soil as shown in Figure 9.

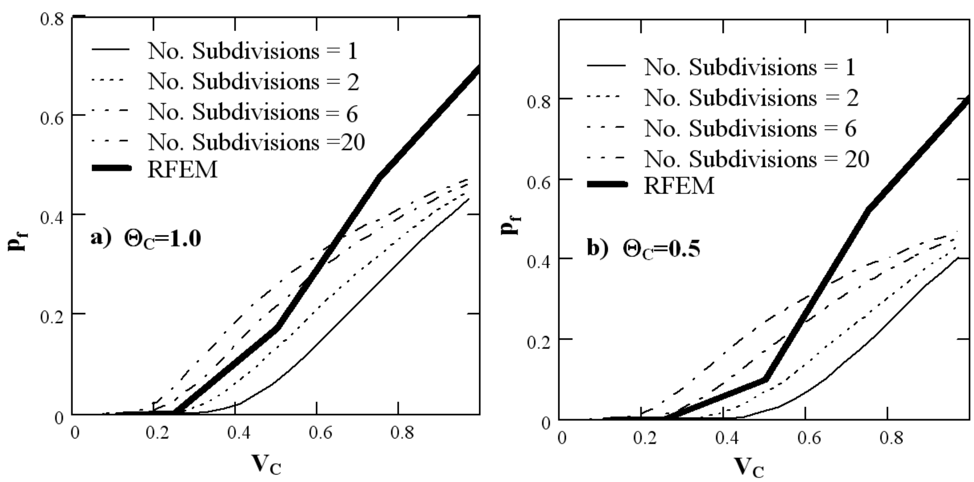

Fig. 11. The probability of failure versus coefficient of variation; the mean is fixed at μ_C=0.25, a) Θ_C=1.0 and b) Θ_C=0.5.

CONCLUDING REMARKS

This paper has demonstrated two methods by which soil variability including spatial correlation can be accounted for in probabilistic slope stability analysis.

In the first method, the critical failure surface was assumed to be circular as would be obtained from classical slope stability considerations. The circle was then subdivided and 1D local averaging performed over a typical arc length. The resulting locally averaged mean and standard deviation were then used in a FOSM analysis. It was found that the number of subdivisions used prior to local averaging had a significant influence on the estimated probability of failure. In particular, if too few subdivisions were used, unconservative predictions were made which underestimated the probability of failure.

A second approach using the rigorous Random Finite Element Method (RFEM) highlighted important deficiencies in the simple approach. Firstly, the RFEM makes no *a priori* assumption about the shape or location of the critical failure surface and allows the slope to “fail where it wants to fail”. In each Monte-Carlo simulation, the analysis is able to “seek out” the critical failure mechanism which can be quite different to the conventional circular path given by classical slope stability methods. This was demonstrated clearly in plots of the plastic shear strain invariant at slope failure for typical realizations of the Monte-Carlo process.

While the probability of failure predicted by the first method was improved by using more subdivisions, methods which fix the failure surface in advance should be avoided. This is because even when local averaging and spatial correlation are properly accounted for, fixed failure surface methods are inherently unconservative because they lead to upper-bound solutions.

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slope stability, probabilistic analysis, finite element analysis, local averaging, spatial correlation.