PROBABILISTIC SETTLEMENT ANALYSIS BY STOCHASTIC AND RANDOM FINITE ELEMENT METHODS

D.V. Griffiths[1] and Gordon A. Fenton[2]

[1]F. ASCE, Geomechanics Research Center, Division of Engineering,
Colorado School of Mines, Golden, Colorado 80401-1887, USA;
email: D.V.Griffiths@mines.edu
[2]M. ASCE, Department of Engineering Mathematics, Dalhousie University, Halifax, Nova Scotia, Canada; email: Gordon.Fenton@dal.ca

ABSTRACT

The paper investigates how statistically described soil stiffness parameters affect the statistics of the settlement of a rigid strip footing, with the aim of estimating its probability of excessive settlement. The paper compares results obtained using the approximate Stochastic Finite Element Method (SFEM) based on First Order Second Moment (FOSM) principles, with the rigorous Random Finite Element Method (RFEM) based on Monte-Carlo simulations. In the SFEM, the paper describes an analytical approach for obtaining derivatives of footing settlement with respect to soil element stiffness. Finally, the paper demonstrates the range of input soil stiffness parameters for which the SFEM gives reasonable results before the first order assumptions start to break down.

SUMARIO

En este trabajo se presenta como los parámetros de rigidez del suelo afectan las estadísticas de asentamiento de zapatas corridas rígidas, con el objeto de estimar la probabilidad de asentamientos excesivos.En este trabajo se comparan los resultados obtenidos usando un método aproximado como lo es el Método Estocástico de Elementos Finitos (SFEM del nombre del método en ingles) basado en el Segundo Momento de Primer Orden (FOSM del nombre en ingles), con el riguroso Método de Elementos Finitos Aleatorios (RFEM del nombre del método en ingles) basado en simulaciones de Monte-Carlo. Este trabajo describe el SFEM como un método analítico para obtener las derivadas del asentamiento de zapatas con respecto a la

rigidez de los elementos del suelo. Finalmente, este trabajo demuestra el rango de valores de rigidez del suelo para el cual el método SFEM da resultados razonables antes de que las suposiciones de primer orden empiezan a no cumplirse.

INTRODUCTION

First order methods such as the First Order Second Moment (FOSM) and the First Order Reliability Method (FORM) have received significant exposure (e.g. Low 1997, Nadim 2002, Duncan, Baecher and Christian) in recent years as relatively simple methods for estimating the probability of events occurring in geotechnical analysis. The basic objective is as follows: given statistical data (mean and standard deviation) for key geotechnical input parameters (e.g. strength parameters c' and $\tan\phi'$, seepage parameters k, settlement parameters E) what are the statistics (mean and standard deviation) of the key output quantities (e.g. Factor of Safety FS, flow rate Q, settlement δ). In the case of the output parameter, if these statistics are combined with an assumed probability density function, the probability of events such as slope failure, excessive flow rates, excessive settlement, etc. can then be estimated.

While these methods are relatively easy to implement and give useful qualitative and sensitivity information about the input and output parameters, they are based on an underlying assumption of a Taylor Series truncated after the linear terms—hence “first order”. In this paper we take a problem of elastic foundation settlement, and compare the approximate FOSM method, as implemented in the stochastic finite element (SFEM) method with a rigorous approach to the same problem using the Random Finite Element Method (RFEM). The RFEM includes no such approximations relating to the Taylor’s series and represents the state-of-the-art in probabilistic geotechnical analysis.

The fundamental problem considered in this paper is shown in Figure 1. input parameters consists of the statistics (mean μ_E, standard deviation σ_E and spatial correlation length θ_E) of Young’s modulus with a constant Poisson’s ratio. The values to be predicted and compared by the two methods under consideration in this paper are the mean μ_δ and standard deviation σ_δ of the settlement.
The paper will identify the limits on input variability for which the FOSM gives reliable estimates of output statistics. Once this range has been established, guidelines can then be produced to help users decide when more sophisticated methods such as RFEM are warranted.

BRIEF REVIEW OF THE FOSM FOR MULTIPLE RANDOM VARIABLES

Consider a function $f(X_1, X_2, \ldots, X_n)$ of n correlated random variables. To a first order of accuracy the mean of the function is given by

$$\mu_f = E\left[(X_1, X_2, \mathrm{L}\ \ , X_n)\right] \approx f(\mu_{X_1}, \mu_{X_2}, \mathrm{K}\ , \mu_{X_n}) \quad (1)$$

and the variance of the function as

$$\sigma_f^2 = \mathrm{Var}\left[(X_1, X_2, \mathrm{L}\ \ , X_n)\right] \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial f}{\partial x_j}\right)\mathrm{Cov}\left[X_i, X_j\right] \quad (2)$$

The first derivatives in (2) are computed at the mean values $(\mu_{X_1}, \mu_{X_2}, \mathrm{K}\ , \mu_{X_n})$ and can be evaluated numerically or, if a functional form exists, analytically.

STOCHASTIC FINITE ELEMENT METHOD (SFEM) USING FOSM

In this paper, a finite element implementation of equations (1) and (2) based on the freely available software of Smith and Griffiths (2004) (1.3) (1.4) will be presented in an elastic settlement analysis where

$$\delta = f(E_1, E_2, \mathrm{K}\ , E_n) \quad (3)$$

$$\mu_\delta = E\left[(E_1, E_2, \mathrm{L}\ \ , E_n)\right] \approx f(\mu_{E_1}, \mu_{E_2}, \mathrm{K}\ , \mu_{E_n}) \quad (4)$$

$$\sigma_\delta^2 = \mathrm{Var}\left[(E_1, E_2, \mathrm{L}\ \ , E_n)\right] \approx \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{\partial \delta}{\partial E_i}\right)\left(\frac{\partial \delta}{\partial E_j}\right)\mathrm{Cov}\left[E_i, E_j\right] \quad (5)$$

The random variables in this case are now the values of Young's modulus assigned to each element in the finite element mesh consisting of square 4-node plane strain elements shown in Figure 2. For illustrative purposes the figure shows a very coarse mesh of 18 elements, but the case studies shown later will use a greater mesh refinement.

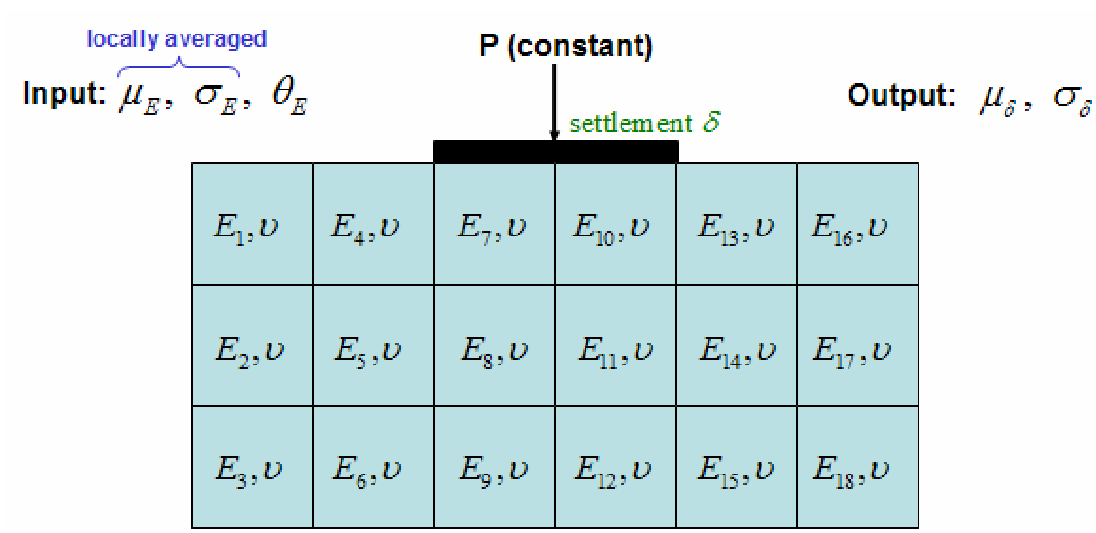


Figure 1. Benchmark elastic settlement problem of a rigid strip footing on soil with spatially random stiffness.

In order to be consistent with the rigorous RFEM to come later, the input "point" statistics (μ_E, σ_E) have been adjusted to account for local averaging due to element size. If we define the side length of the square finite elements as $\alpha\theta$, the local averaging is expressed through a variance reduction factor γ defined

$$\gamma = \frac{4}{\alpha^4} \int_0^\alpha \int_0^\alpha (\alpha - x)(\alpha - y) e^{-2\sqrt{x^2+y^2}} \, dx \, dy \tag{6}$$

When assuming a lognormal distribution of E as we have done in this study, local averaging causes both the mean and standard deviation to be reduced (see e.g. Griffiths and Fenton 2004). Essentially, local averaging becomes more significant when α is large and less significant when α is small.

The mean settlement from equation (4) is easily computed by a single finite element analysis as shown in equation (8) (with all E values set to the mean), but in order to compute the variance from equations (5) the covariance and the derivatives must first be found.

Covariance

The covariance between any two elements is stored in a "Covariance Matrix" given by

$$\text{Cov}[E_i, E_j] = \rho \, \sigma_E^2 \tag{7}$$

where $\rho = e^{-2\tau/\theta}$ and τ is the centroidal distance between element ι and element j.

Derivatives

The basic stiffness relationship for the mesh is given by

$$\begin{aligned} &[\mathbf{K}]\{\boldsymbol{\delta}\} = \{\mathbf{F}\} \text{ where} \\ &[\mathbf{K}] = f(\mu_{E_1}, \mu_{E2}, \ldots, \mu_{E_n}) \end{aligned} \tag{8}$$

hence for each random variable $E_i, \; i = 1, 2, \ldots, n$

$$\left(\frac{\partial}{\partial E_i}[\mathbf{K}]\right)\{\boldsymbol{\delta}\} + [\mathbf{K}]\frac{\partial}{\partial E_i}\{\boldsymbol{\delta}\} = \frac{\partial}{\partial E_i}\{\mathbf{F}\} \tag{9}$$

where $[\mathbf{K}]$ and $\{\boldsymbol{\delta}\}$ are evaluated using μ_E. For constant loading

$$\frac{\partial}{\partial E_i}\{\mathbf{F}\} = \{\mathbf{0}\} \tag{10}$$

hence, $$[\mathbf{K}]\frac{\partial}{\partial E_i}\{\boldsymbol{\delta}\} = -\left(\frac{\partial}{\partial E_i}[\mathbf{K}]\right)\{\boldsymbol{\delta}\} \tag{11}$$

thus, all the required derivatives can be obtained by solution of a single set of equations (n times). The right-hand-side is obtained by assembling all the element derivative matrices which for the square 4-node elements uses in this study can be obtained in closed form.

The analyses described by equations (8) and (11) involve $n+1$ equation solutions, where n is the number of elements in the mesh. Since the left-hand-side matrix $[\mathbf{K}]$ is the same in all cases it need only be factorized once, followed by $n+1$ forward and back-substitutions.

The mesh used in the current study is shown in Figure 2 and has 840 elements, thus each parametric combination considered involved one matrix factorization followed by 841 forward and back-substitutions. The footing load is held constant at $P = 1$.

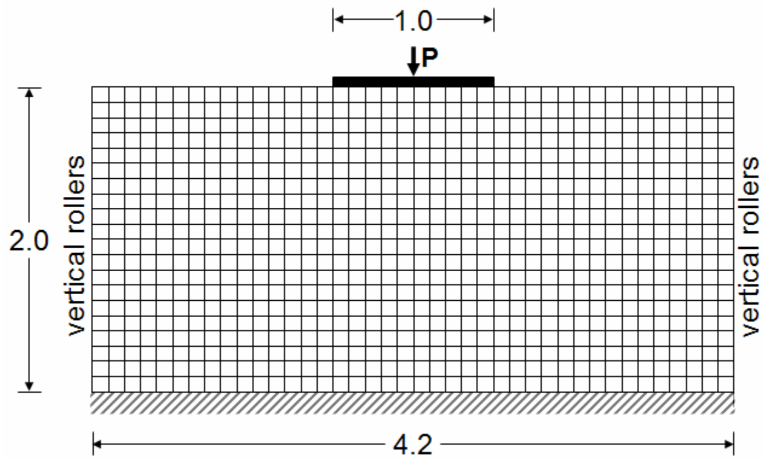


Figure 2. Mesh used for SFEM vs. RFEM study of elastic settlement of a rigid strip footing on soil with spatially random stiffness.

REVIEW OF THE RANDOM FINITE ELEMENT METHOD (RFEM)

The Random Finite Element Method first described by Griffiths and Fenton (1993) and Fenton and Griffiths (1993) involves generating a random field of soil properties with controlled first and second moment statistics, which are then mapped onto a finite element mesh. A conventional elastic finite element analysis using these properties is then performed, after which the process is repeated many times using Monte-Carlo simulations. For each realization of the Monte-Carlo process the underlying statistics of Young's modulus are held constant, however the spatial distribution is different and the computed settlement of the footing under a constant load is different each time. Following a sufficient number of repetitions the output values become statistically stable and can be analyzed. It should be noted that unlike the SFEM analysis, this method gives a histogram of settlement values which can be fitted to an appropriate function (e.g. lognormal) for probabilistic interpretation. In the current study 1000 realizations were used for each parametric combination. The program used in this study and many others for performing geotechnical analysis by RFEM are freely available from web site http://www.engmath.dal.ca/rfem

The random field is generated using the Local Average Subdivision (LAS) method (Fenton and Vanmarcke 1990) which takes full account of local averaging as described previously. A convenient aspect of LAS is that the random field is generated over cells that are the same size as the finite elements greatly facilitating the mapping of properties onto elements. A typical result from an RFEM analysis is shown in Figure 3.

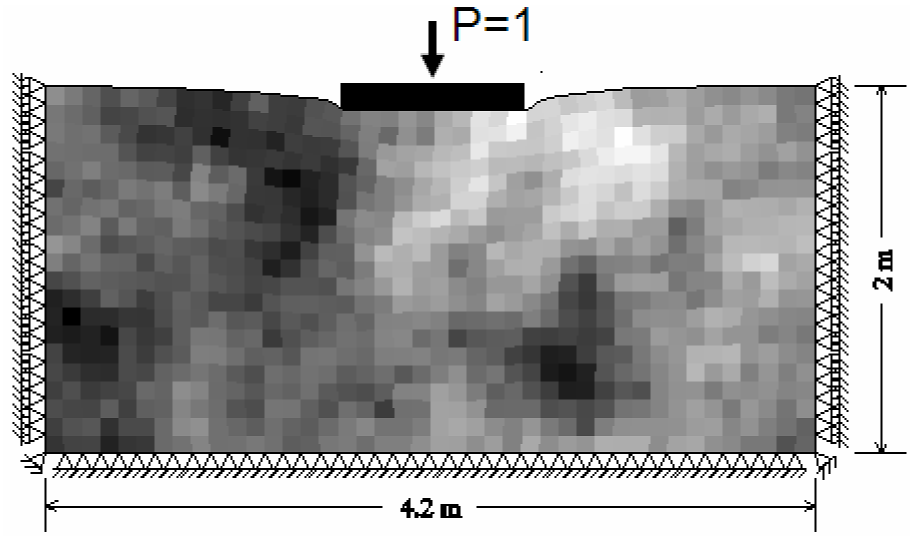


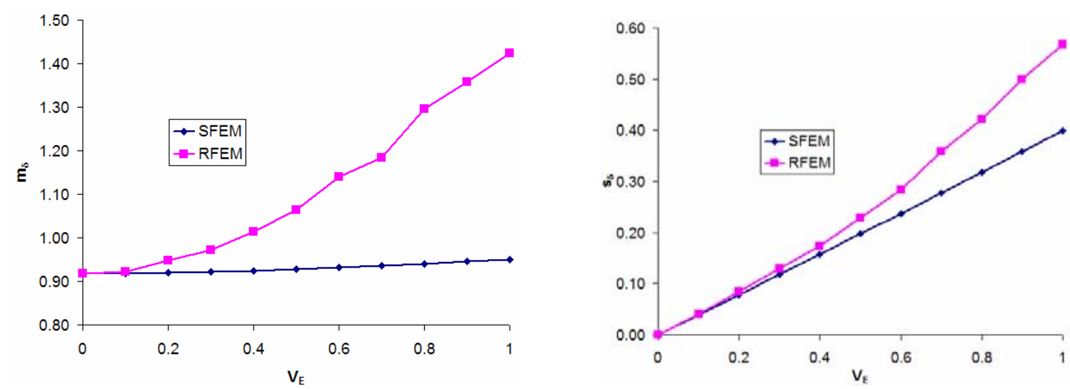
Figure 3. Typical RFEM analysis showing the random field of Young's modulus (darker is stiffer).

RESULTS AND DISCUSSION

Parametric studies (e.g. Herlyck 2005) based on the mesh shown in Figure 2 involved the following ranges of values:

$$\mu_E = 1, \quad 0.0 < \sigma_E < 1,$$
$$0 < \theta_E < 5, \quad \upsilon = 0.25$$

In the plots that follow, the Coefficient of Variation of Young's modulus, $V_E = \sigma_E / \mu_E$ has been used on the abscissa.



Figures 4. Variation of (a) mean and (b) standard deviation of footing settlement vs. coefficient of variation of input Young's modulus.

The results displayed in Figures 4-6 were obtained with the spatial correlation length θ_E fixed to 1.0. Figures 4(a) and (b) show the variation of the computed mean (m_δ) and standard deviation (s_δ) of footing settlement by both methods.

The mean (m_δ) and standard deviation (s_δ) of settlement is clearly underestimated by SFEM as compared with RFEM with the difference growing consistently with the input coefficient of variation of Young's modulus (V_E). Matthies *et al.* 1997 observed a similar divergence between Monte-Carlo and "perturbation" methods in a quite different application. The percentage difference between SFEM and RFEM is similar in both plots as shown in Figure 5, with the error in the SFEM results growing to about 13% for V_E =0.5, which might be considered an upper-bound on stiffness variability for many soils. The consistency of the error in the mean and standard deviation is further emphasized in Figure 6, where the coefficient of variation $(V_\delta = s_\delta / m_\delta)$ of the predicted settlement by both methods shows remarkable agreement.

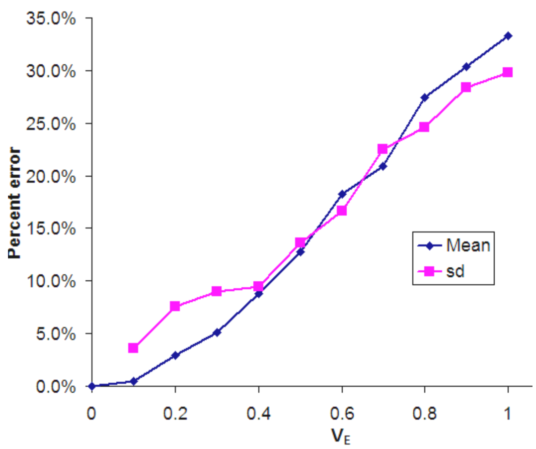


Figure 5. Error in SFEM vs. V_E .

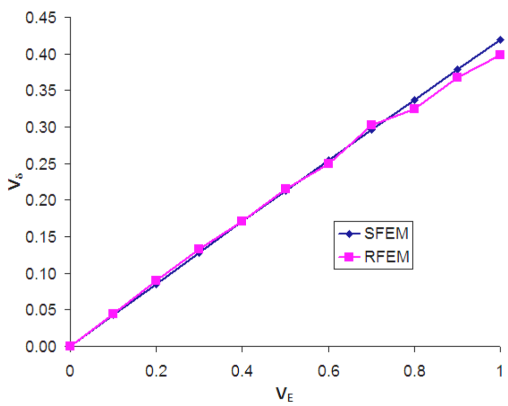
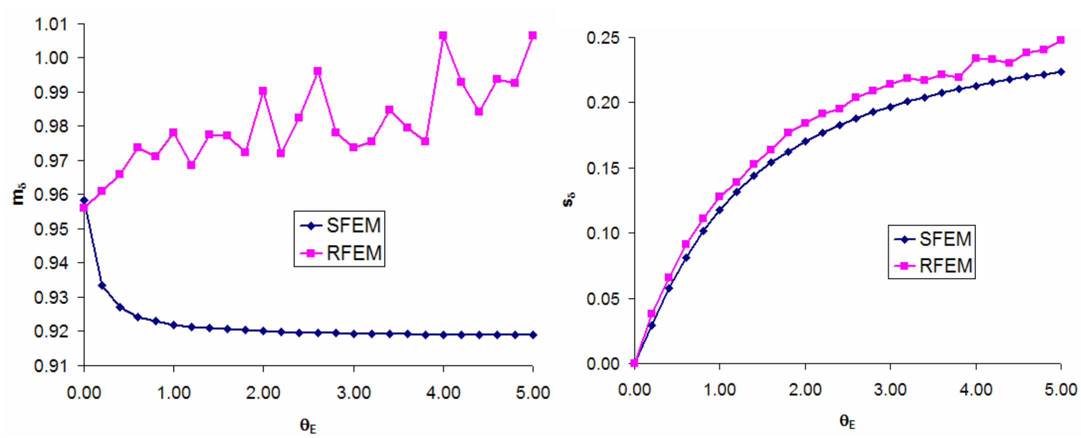


Figure 6. V_δ vs. V_E



Figures 6. Variation of (a) mean and (b) standard deviation of footing settlement vs. spatial correlation length Young's modulus.

The next set of results investigate the influence on the spatial correlation length θ_E. In these cases the coefficient of variation of the input Young's modulus has been fixed to $V_E = 0.3$. Figures 6(a) and (b) show the variation of the computed mean (m_δ) and standard deviation (s_δ) of footing settlement respectively, by both methods for θ_E in the range $0 \le \theta_E \le 5$. As $\theta_E \to 0$ the local averaging of Young's modulus is maximized and two limiting conditions are approached: (i) the mean settlement in both cases tends to a value corresponding to a homogeneous soil with stiffness given by the median Young's modulus, which in this case equals 0.96, (ii) the standard deviation in both cases tends to zero.

As θ_E increases, the locally averaging effect is reduced and both the mean and standard deviation of Young's modulus increase towards the "point" values as shown in Figure 6(a). In the SFEM analyses, this is clearly evident in the reduced mean settlement, which as θ_E increases, tends to about 0.92, which is the value that would be given in a homogeneous soil with $E = 1$. The RFEM results are more ragged due to the Monte-Carlo simulations, however there is a clear trend of gradually increasing mean settlement which may at first seen initially counter-intuitive, since the mean Young's modulus is also increasing. The explanation lies in the fact that as θ_E increases, the standard deviation of Young's modulus also increases, meaning both higher and lower stiffness values are being assigned to the elements below the footing. The lower stiffness elements dominate the solution however, and more than compensate for the increased mean stiffness. The standard deviation of settlement in both cases also increases with increased θ_E as shown in Figure 6(b). Theoretically, the standard deviation is tending towards a limiting value (as $\theta_E \to \infty$) of about 0.28 corresponding to a coefficient of variation of $V_\delta \approx V_E = 0.3$.

PROBABILISTIC INTERPRETATION

While comparisons of the predicted mean and standard deviation of settlement have been the focus in the previous sections, the usual goal of analyses such as SFEM and RFEM is to predict the *probability* that the settlement exceeds some design criterion.

In the following comparison we take the case where the soil has properties given by $\mu_E = 1, \sigma_E = 0.5 \ (V_E = 0.5)$, and $\theta_E = 1$ which might be considered typical values of soil stiffness (see e.g. Lee *et al.* 1983). For this particular case, the computed parameters by the two methods are summarized in Table 1.

Table 1. Footing settlement statistics computed by SFEM and RFEM

Method	$\mathbf{m_\delta}$	$\mathbf{s_\delta}$	$\mathbf{V_\delta}$
SFEM	0.928	0.197	0.212
RFEM	1.064	0.228	0.215

To illustrate the process, let us assume that the distribution of settlement (like Young's modulus) is lognormally distributed, and then estimate the probability that the footing settlement exceeds δ_{design}.

First we need to compute the statistics of the underlying normal distribution of $\ln\delta$ of using the formulas:

$$m_{\ln\delta} = \ln m_\delta - \frac{1}{2}\ln\left\{1+V_\delta^{\,2}\right\}$$
$$s_{\ln\delta} = \sqrt{\ln\left\{1+V_\delta^{\,2}\right\}} \qquad (12)$$

leading to the values given in Table 2.

Table 2. Parameters of underlying normal distribution of $\ln\delta$

Method	$m_{\ln\delta}$	$s_{\ln\delta}$
SFEM	-0.097	0.210
RFEM	0.040	0.212

The required probability is then given by

$$P[\delta > \delta_{\text{design}}] = 1 - P[\delta \le \delta_{\text{design}}]$$
$$= 1 - \Phi\left(\frac{\ln(\delta_{\text{design}}) - m_{\ln\delta}}{s_{\ln\delta}}\right) \qquad (13)$$

where $\Phi(.)$ is the Cumulative Standard Normal function which can be obtained from standard tables.

If we let $\delta_{\text{design}} = 1.4$, then in the case of the SFEM analysis we get

$$P[\delta > \delta_{\text{design}}]$$
$$= 1 - \Phi\left(\frac{\ln(1.4) + 0.097}{0.210}\right) \qquad (14)$$
$$= 1 - \Phi(2.06)$$
$$= 1 - 0.98 = 0.02\,(2\%)$$

and for RFEM

$$P[\delta > \delta_{\text{design}}]$$
$$= 1 - \Phi\left(\frac{\ln(1.4) - 0.04}{0.212}\right) \qquad (15)$$
$$= 1 - \Phi(1.40)$$
$$= 1 - 0.93 = 0.07\,(7\%)$$

The SFEM results underestimate the probability of design failure and are on the “unsafe” side. This conclusion would have been reached for any initial choice of θ_E and V_E. The lower probability of failure is clearly due to the lower mean settlements consistently predicted by SFEM, but fundamentally, the shortcomings of SFEM as implemented in this paper is that unlike the RFEM, it is unable to compute the influence of a spatially variable soil stiffness.

CONCLUDING REMARKS

The paper has compared the performance of a Stochastic Finite Element Method based on First Order Second Moment assumptions with the Random Finite Element Method in a probabilistic study of foundation settlement. The analyses considered both the coefficient of variation V_E variance and spatial correlation length θ_E of the input Young’s modulus. While holding θ_E constant, the computed mean and standard deviation of the settlement by both methods were similar for low input V_E, but diverged quite rapidly as V_E increased. The coefficient of variation of the settlement V_δ however, was in remarkably good agreement by both methods even for high V_E.

A key difference between the methods was highlighted when V_E was held constant and the spatial correlation length θ_E gradually increased. In this case the mean settlements went in opposite directions with the SFEM mean settlement falling to reflect the rising locally averaged mean Young's modulus, and the RFEM mean settlement slowly rising. Even though the locally averaged mean Young's modulus is also rising in the RFEM, the rise in mean settlement is explained by the fact that Monte-Carlo based RFEM is modeling a spatially random material in which the less stiff elements are having a bigger influence on the overall settlement than the stiffer elements and overcompensating for the increased mean stiffness. The SFEM has no direct way of modeling the real influence of spatial variability. In all parametric combination considered, the mean and standard deviation of settlement predicted by SFEM were smaller than those predicted by RFEM.

In conclusion, care must be taken when using the less rigorous SFEM approach in probabilistic foundation settlement analysis While SFEM may give reasonable predictions for low input property variance, it will always lead to underestimates of the probability of design failure as compared with the state-of-the-art RFEM.

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REFERENCES

Baecher, G. B. and Christian, J. (2003) "Reliability and statistics in geotechnical engineering." John Wiley & Sons, New York.

Duncan, J. M. (2000) "Factors of safety and reliability in geotechnical engineering." *J Geotech Geoenviron*, vol.126, no.4, pp.307-316.

Fenton, G.A. and Vanmarcke, E.H. (1990) "Simulation of random fields via local average subdivision." *J Eng Mech*, vol.116, no.8, pp.1733-1749.

Fenton, G.A. and Griffiths, D.V. (1993) "Statistics of block conductivity through a simple bounded stochastic medium." *Water Resour Res*, vol.29, no.6, pp.1825-1830.

Fenton, G.A. and Griffiths, D.V. (2005). "Three-dimensional probabilistic foundation settlement" *J Geotech Geoenviron*, vol.131, no.2, pp.232-239

Griffiths, D.V. and Fenton, G.A. (1993) "Seepage beneath water retaining structures founded on spatially random soil." *Géotechnique*, 43, no.4, pp.577-587.

Herlyck, J.A. (2005) "A comparison of First Order Second Moment and Monte Carlo programs in determining rigid foundation elastic settlement." REU Independent Study, Colorado School of Mines.

Lee, I.K., White, W. and Ingles, O.G. (1983). "*Geotechnical engineering*", Pitman, London.

Low, B.K., and Tang, W.H. (1997). "Efficient reliability evaluation using spreadsheet." *Journal of EngineeringMechanics*, ASCE, 123(7), 749-752.

Matthies, H.G., Brenner, C.E., Bucher, C.G. and Soares, C.G. (1997) "Uncertainties in probabilistic numerical analysis of structures and solids -- stochastic finite elements." *J Structural Safety*, vol.19, no.3, pp.283-336.

Nadim, F. (2002) "Probabilistic methods for geohazard problems: State-of-the-art", In *Probabilistics in Geotechnics: Technical and Economic Risk estimation*, eds. R. Pöettler *et al.*, pub.VGE, Essen, Germany, pp.333-350.

Smith, I.M. and Griffiths, D.V. (2004), "Programming the finite element method." 4th ed., John Wiley & Sons, Chichester.