Load and Resistance Factor Design of Strip Footings

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Abstract

This paper proposes a Load and Resistance Factor Design (LRFD) approach for the bearing capacity design of a strip footing. The load factors used are as specified by the National Building Code of Canada. The resistance factors required to achieve a certain acceptable failure probability are estimated as a function of the spatial variability of the soil as well as by the level of "understanding" of the soil properties in the vicinity of the foundation. The analytical results are validated by simulation. The results are primarily intended to aid the development of the next generation of reliability-based geotechnical design codes, but can also be used to guide current designs.

Introduction

Design of a shallow footing typically begins with a site investigation aimed at determining the strength of the founding soil or rock. Once this information has been gathered, the geotechnical engineer is in a position to determine the footing dimensions required to avoid entering various limit states. The limit states that are usually considered in the footing design are serviceability limit states (typically deformation) and ultimate limit states. The latter is concerned with safety and includes the load-carrying capacity, or *bearing capacity*, of the footing.

This paper develops a load and resistance factor design (LRFD) approach for strip footings designed against bearing capacity failure. The design goal is to determine the footing dimensions such that the *resistance* to the load, R_u, satisfies

$$\phi_g R_u \geq \sum_i \alpha_i L_{i_c} \tag{1}$$

where ϕ_g is the *geotechnical resistance factor*, R_u is the *ultimate geotechnical resistance*, I is the *importance factor*, α_i is the i'th *load factor*, and L_{i_c} is the i'th *characteristic load effect*. The goal of this paper is to determine the relationship between ϕ_g and the probability that the designed footing will experience a bearing capacity failure.

The ultimate geotechnical resistance, R_u, is determined using characteristic soil properties, in this case characteristic values of the soil's cohesion, c, and friction angle, ϕ. Only one load combination will be considered in this paper, $\alpha_L L_{L_c} + \alpha_D L_{D_c}$, where $L_{L_c} = k_L \mu_L$ is the characteristic live load defined as a bias factor, $k_L = 1.41$ (Allen, 1975), times the mean live load, μ_L, and $L_{D_c} = k_D \mu_D$ is the characteristic dead load, similarly defined as a bias factor $k_D = 1.18$ (Becker, 1996), times the mean dead load, μ_D. The live and dead load factors, $\alpha_L = 1.5$ and $\alpha_D = 1.25$, respectively are as specified by the National Building Code of Canada (NBCC, 2006).

To determine the resistance factor, ϕ_g, required to achieve a certain acceptable reliability of the constructed footing against bearing failure, the founding soil will be modeled as a 2-D random field and the design process involves first taking a series of m soil samples are over depth at a single location a distance r from the footing center (as in a CPT or SPT sounding). The characteristic cohesion, $\hat{c}$, and characteristic friction angle, $\hat{\phi}$, are computed from the observations (denoted by a superscript o) as follows,

$$\hat{c} = \exp\left\{ \frac{1}{m} \sum_{i=1}^{m} \ln c_i^o \right\}, \qquad \hat{\phi} = \frac{1}{m} \sum_{i=1}^{m} \phi_i^o \tag{2}$$

The soil will be assumed weightless so that the characteristic ultimate bearing stress, $\hat{q}_u$, simplifies to

$$\hat{q}_u = \hat{c}\hat{N}_c \tag{3}$$

where $\hat{N}_c$ is the characteristic bearing capacity factor

$$\hat{N}_c = \frac{e^{\pi \tan \hat{\phi}} \tan^2\left(\frac{\pi}{4} + \frac{\hat{\phi}}{2}\right) - 1}{\tan \hat{\phi}} \tag{4}$$

Since $R_u = B\hat{q}_u$, where B is the footing width, Eq. 1 can be solved for the required footing width

$$B = \frac{I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]}{\phi_g \hat{q}_u} = \frac{I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]}{\phi_g \hat{c}\hat{N}_c} \tag{5}$$

The design philosophy is to find the required footing width B such that the probability that the actual load, L, exceeds the actual resistance, $q_u B$, is less than some small acceptable failure probability, p_m. If p_f is the actual failure probability, then

$$p_f = \mathrm{P}\left[L > q_u B\right] = \mathrm{P}\left[L > \bar{c}\bar{N}_c B\right] \tag{6}$$

where $q_u = \bar{c}\bar{N}_c$ and $\bar{c}$ and $\bar{N}_c$ are those effective *uniform* soil properties which give the same bearing capacity as the actual spatially variable soil. The value of $\bar{N}_c$ is obtained by using an effective friction angle $\bar{\phi}$ in Eq. 4. A sucessful design methodology will have $p_f \le p_m$. Substituting Eq. 5 into Eq. 6 and collecting random terms to the left of the inequality leads to

$$p_f = \mathrm{P}\left[L \frac{\hat{c}\hat{N}_c}{\bar{c}\bar{N}_c} > \frac{I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]}{\phi_g}\right] \tag{7}$$

Letting $Y = L(\hat{c}\hat{N}_c)/(\bar{c}\bar{N}_c)$ means that

$$p_f = \mathrm{P}\left[Y > \frac{I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]}{\phi_g}\right] \tag{8}$$

and the task is to find the distribution of Y. Assuming that Y is lognormally distributed (an assumption found to be reasonable by Fenton et al., 2007, and which is also supported to some extent by the central limit theorem), then

$$\ln Y = \ln L + \ln \hat{c} + \ln \hat{N}_c - \ln \bar{c} - \ln \bar{N}_c \tag{9}$$

is normally distributed and p_f can be found once the mean and variance of $\ln Y$ are determined. The mean of $\ln Y$ is

$$\mu_{\ln Y} = \mu_{\ln L} + \mu_{\ln \hat{c}} + \mu_{\ln \hat{N}_c} - \mu_{\ln \bar{c}} - \mu_{\ln \bar{N}_c} \tag{10}$$

and the variance of $\ln Y$ is

$$\sigma^2_{\ln Y} = \sigma^2_{\ln L} + \sigma^2_{\ln \hat{c}} + \sigma^2_{\ln \bar{c}} + \sigma^2_{\ln \hat{N}_c} + \sigma^2_{\ln \bar{N}_c} - 2\text{Cov}\left[\ln \bar{c}, \ln \hat{c}\right] - 2\text{Cov}\left[\ln \bar{N}_c, \ln \hat{N}_c\right] \tag{11}$$

where the load, L, and soil properties, c and ϕ have been assumed mutually independent.

Analytical approximation to the probability of failure

To find the terms in Eq's 10 and 11, it is assumed that the effective cohesion, $\bar{c}$, is a geometric average over a domain of size $D = W \times W$ immediately under the footing (see Figure 1). Similarly, the effective friction angle is assumed to be an arithmetic average over the same domain;

$$\bar{c} = \exp\left\{ \frac{1}{D} \int_D \ln c(\underset{\sim}{x})\, d\underset{\sim}{x} \right\}, \qquad \bar{\phi} = \frac{1}{D} \int_D \phi(\underset{\sim}{x})\, d\underset{\sim}{x} \tag{12}$$

The dimension W was found by trial and error to be best approximated as 40% of the average mean wedge zone depth,

$$W = \frac{0.4}{2} \hat{\mu}_B \tan\left(\frac{\pi}{4} + \frac{\mu_\phi}{2}\right) \tag{13}$$

where μ_ϕ is the mean friction angle (in radians), within the zone of influence of the footing, and $\hat{\mu}_B$ is an estimate of the mean footing width obtained by using mean soil properties (μ_c and μ_ϕ) in Eq. 5. To first order, the mean of N_c is,

$$\mu_{N_c} \simeq \frac{e^{\pi \tan \mu_\phi} \tan^2\left(\frac{\pi}{4} + \frac{\mu_\phi}{2}\right) - 1}{\tan \mu_\phi} \tag{14}$$

Armed with the above information and assumptions, the components of Eq's 10 and 11 can be computed as follows given the basic statistical parameters of the loads, c, and ϕ, the number and locations of the soil samples, and the averaging domain size D.

Assuming that the total load L is equal to the sum of live and dead loads, i.e. $L = L_{L_e} + L_D$, both of which are random, then

$$\mu_{\ln L} = \ln(\mu_L) - \tfrac{1}{2} \ln\left(1 + V_L^2\right), \qquad \sigma^2_{\ln L} = \ln\left(1 + V_L^2\right) \tag{15}$$

where $\mu_L = \mu_L + \mu_D$ is the sum of the mean live and (static) dead loads, and V_L is the coefficent of variation of the total load defined by

$$V_L^2 = \frac{\sigma^2_{L_e} + \sigma^2_D}{\mu_{L_e} + \mu_D} \tag{16}$$

With reference to Eq. 2,

$$\mu_{\ln \hat{c}} = \mu_{\ln c}, \qquad \sigma^2_{\ln \hat{c}} = \sigma^2_{\ln c}\gamma(\Delta x, H) \tag{17}$$

where γ is the variance function defined by

$$\gamma(D_1, D_2) = \frac{4}{(D_1 D_2)^2}\int_0^{D_1}\int_0^{D_2}(D_1-\tau_1)(D_2-\tau_2)\rho(\tau_1,\tau_2)\,d\tau_1\,d\tau_2 \tag{18}$$

Similarly, with reference to Eq. 12,

$$\mu_{\ln \bar{c}} = \mu_{\ln c} \qquad \sigma^2_{\ln \bar{c}} = \sigma^2_{\ln c}\gamma(W, W) \tag{19}$$

Since $\mu_{\hat{\phi}} = \mu_\phi$ (see Eq. 2), the mean and variance of $\hat{N}_c$ can be obtained using first order approximations to expectations of Eq. 4 (Fenton et al., 2003), as follows,

$$\mu_{\ln \hat{N}_c} = \mu_{\ln N_c} \simeq \ln \frac{e^{\pi \tan \mu_\phi}\tan^2\left(\frac{\pi}{4}+\frac{\mu_\phi}{2}\right)-1}{\tan \mu_\phi} \tag{20}$$

$$\sigma^2_{\ln \hat{N}_c} \simeq \sigma^2_{\hat{\phi}}\left(\frac{d\ln \hat{N}_c}{d\hat{\phi}}\bigg|_{\mu_\phi}\right)^2 = \sigma^2_{\hat{\phi}}\left[\frac{bd}{bd^2-1}\left[\pi(1+a^2)d+1+d^2\right]-\frac{1+a^2}{a}\right]^2 \tag{21}$$

where $a = \tan(\mu_\phi)$, $b = e^{\pi a}$, $d = \tan\left(\frac{\pi}{4}+\frac{\mu_\phi}{2}\right)$. The variance of $\hat{\phi}$ is given by Fenton et al. (2007) as

$$\sigma^2_{\hat{\phi}} = \sigma^2_\phi\gamma(\Delta x, H), \qquad \sigma_\phi \simeq \frac{0.46(\phi_{max}-\phi_{min})s}{\sqrt{4\pi^2+s^2}} \tag{22}$$

where all angles are measured in radians.

Since $\mu_{\bar{\phi}} = \mu_\phi$ (see Eq. 12), the mean and variance of $\bar{N}_c$ can be obtained in the same fashion as for $\hat{N}_c$ – in fact, they only differ due to differing local averaging in the variance calculation so that $\mu_{\ln \bar{N}_c} = \mu_{\ln N_c}$ and $\sigma^2_{\ln \bar{N}_c}$ is obtained using $\sigma^2_{\bar{\phi}} = \sigma^2_\phi\gamma(W, W)$ in Eq. 21 instead of $\sigma^2_{\hat{\phi}}$.

The covariance between the observed cohesion values and the effective cohesion beneath the footing is $\text{Cov}\,[\ln \bar{c}, \ln \hat{c}] \simeq \sigma^2_{\ln c}\gamma_{DQ}$, where the averaging domains are shown in Figure 1 and

$$\gamma_{DQ} = \frac{1}{(W^2\Delta x H)^2}\int_{-W/2}^{W/2}\int_{H-W}^{H}\int_{r-\Delta x/2}^{r+\Delta x/2}\int_0^H \rho(\xi_1-x_1,\xi_2-x_2)\,d\xi_2\,d\xi_1\,dx_2\,dx_1 \tag{23}$$

which can be evaluated by Gaussian quadrature (see Griffiths and Smith, 2006, for details).

The covariance between $\bar{N}_c$ and $\hat{N}_c$ is similarly approximated by $\text{Cov}\left[\ln \bar{N}_c, \ln \hat{N}_c\right] \simeq \sigma^2_{\ln N_c}\gamma_{DQ}$ where $\sigma^2_{\ln N_c}$ is obtained by using σ^2_ϕ (see Eq. 22) in Eq. 21 instead of $\sigma^2_{\hat{\phi}}$. Substituting these results into Eq's 10 and 11 gives

$$\mu_{\ln Y} = \mu_{\ln L} \tag{24}$$

$$\sigma^2_{\ln Y} = \sigma^2_{\ln L} + \left[\sigma^2_{\ln c}+\sigma^2_{\ln N_c}\right]\left[\gamma(\Delta x, H)+\gamma(W,W)-2\gamma_{DQ}\right] \tag{25}$$

which can now be used in Eq. 8 to produce estimates of p_f. Letting $q = I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]$ the probability of failure becomes

$$p_f = \mathrm{P}\left[Y > q/\phi_g\right] = \mathrm{P}\left[\ln Y > \ln(q/\phi_g)\right] = 1 - \Phi\left(\frac{\ln(q/\phi_g) - \mu_{\ln Y}}{\sigma_{\ln Y}}\right) \qquad (26)$$

where Φ is the standard normal cumulative distribution function.

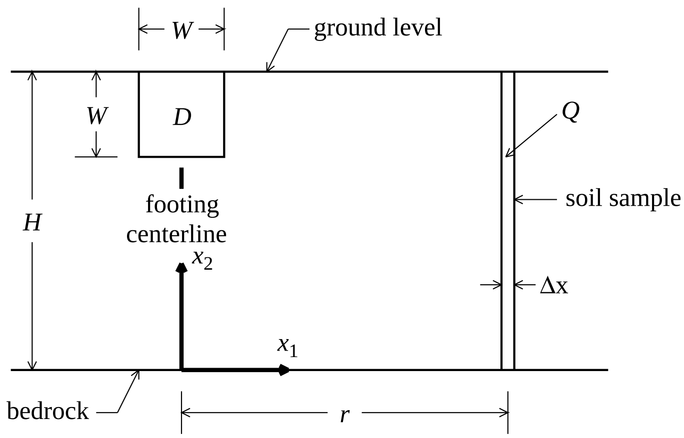

Figure 1. Averaging regions used to predict probability of bearing capacity failure.

Required resistance factor

Eq. 26 can be inverted to find a relationship between the acceptable probability of failure, $p_f = p_m$, and the resistance factor, ϕ_g, required to achieve that acceptable failure probability,

$$\phi_g = \frac{I\left[\alpha_L L_{L_c} + \alpha_D L_{D_c}\right]}{\exp\left\{\mu_{\ln Y} + \sigma_{\ln Y} z_{p_m}\right\}} \qquad (27)$$

where z_{p_m} is the standard normal value which satisfies $\Phi\left(z_{p_m}\right) = 1 - p_m$. For example, if $p_m = 0.001$, then $z_{p_m} = 3.09$.

The following parameters will be varied to investigate their effects on the resistance factor required to achieve a target failure probability p_m;

1) Three values of p_m are considered, 0.01, 0.001, and 0.0001, corresponding to reliability indices of approximately 2.3, 3.1, and 3.7, respectively.
2) The correlation length, θ is varied from 0.0 to 50.0 m.
3) Four coefficients of variation for cohesion are considered, $V_c = 0.1, 0.2, 0.3$, and 0.5. The corresponding coefficients of variation for friction angle are $V_\phi = 0.07, 0.14, 0.20$, and 0.29.
4) Three sampling locations are considered: $r = 0, 4.5$, and 9.0 m from the footing centerline (see Figure 1 for the definition of r).

Figure 2 shows the resistance factors required for three cases; a) sampling directly under the footing ($r = 0$), b) sampling at a distance of 4.5 m, and c) at a distance of 9.0 m from the footing centerline. The worst case correlation length is clearly between about 1 to 5 m, as evidenced by the fact that in all plots the lowest resistance factor occurs when $1 < \theta < 5$ m. This worst case correlation length is of the same magnitude as the footing width ($\hat{\mu}_B = 1.26$ m).

As expected the smallest resistance factors correspond to poorest understanding of the soil properties under the footing (i.e. when the soil is sampled 9 m away from the footing centerline). When the cohesion coefficient of variation is relatively large, V_c = 0.5, with corresponding $V_\phi \simeq$ 0.29, the worst case value of ϕ_g is 0.23 in order to achieve p_m = 0.001. In other words, there will be a significant construction cost penalty if a footing is designed using a low quality site investigation which is unable to reduce the residual variability to less than V_c = 0.5.

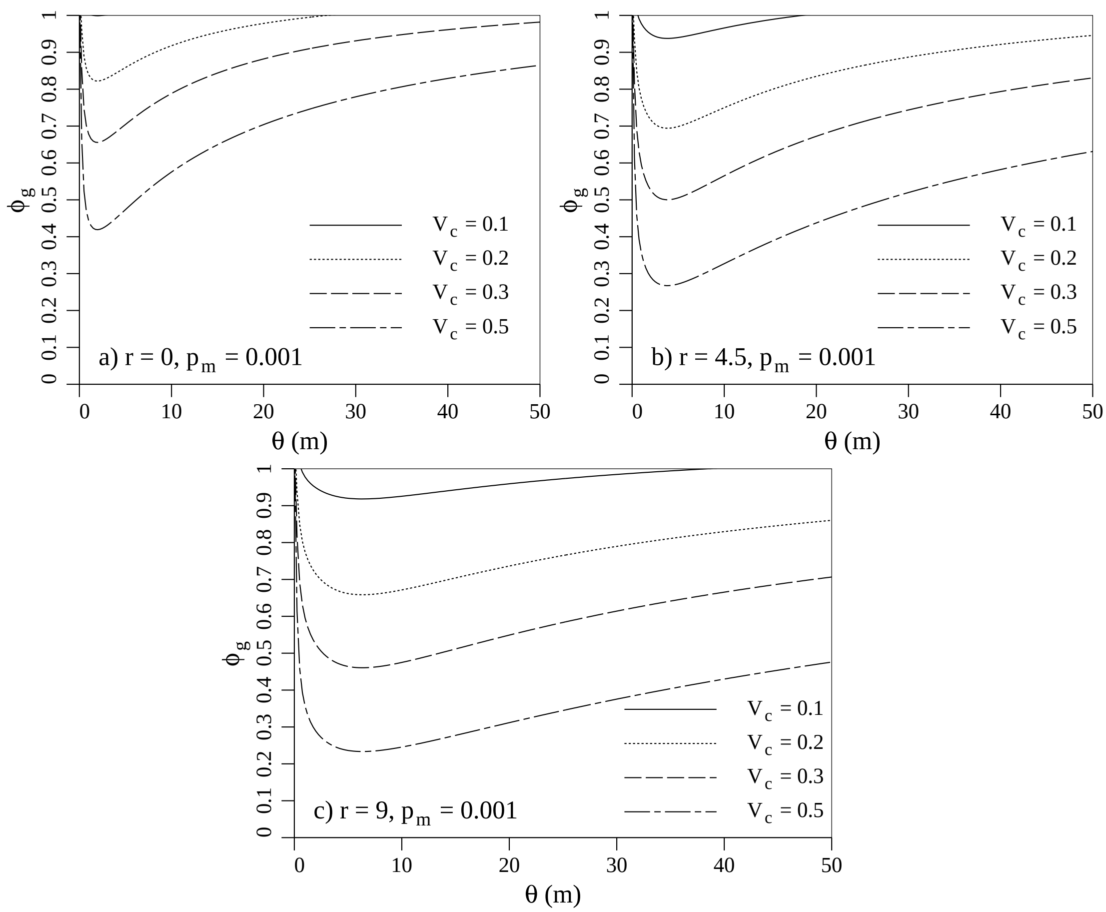

Figure 2. Resistance factors required to achieve acceptable failure probability, p_m = 0.001, when soil is sampled at three different distances, r in m, from the footing centerline.

The "worst case" resistance factors required to achieve the indicated maximum acceptable failure probabilities, as seen in Figure 2, are summarized in Table 1. The Table also includes two other acceptable failure probability values. In the absence of better knowledge about the actual correlation length at the site in question, these factors are the largest values that should be used in the LRFD bearing capacity design of a strip footing.

If the moderate case where V_c = 0.3 and p_m = 0.001, the worst case ϕ_g is 0.66, 0.50, and 0.46 for r = 0, 4.5, and 9.0 m, respectively. Foye et al. (2006) recommend a resistance factor of 0.7 for a similar problem, which agrees quite well with the r = 0 result. The Canadian Foundation Engineering Manual (CFEM, 2006) recommends ϕ_g = 0.5, which agrees with the r = 4.5 result, while the Australian Standard Bridge Design code (2004) recommends ϕ_g = 0.45, which is in very close agreement with the

r = 9.0 result. Possibly the Australian code assumes worse site investigations, or is aimed at a lower acceptable failure probability.

Apparently the resistance factor recommended by Foye et al. (2006) assumes very good site understanding – they specify that the design assumes a CPT investigation which is presumably directly under the footing. Foye's recommended resistance factor is based on a reliability index of β = 3, which corresponds to p_m = 0.0013, which is very close to that used in Table 3 (p_m = 0.001). The small difference between the "current study" r = 0 result and Foye's may be due to differences in load bias factors – these are not specified by Foye et al.

Table 1. Worst case resistance factors for various coefficients of variation, V_c, distance to sampling location, r, and acceptable failure probalities, p_m.

	r = 0.0 m			r = 4.5 m			r = 9.0 m		
V_c	p_m = 0.01	0.001	0.0001	p_m = 0.01	0.001	0.0001	p_m = 0.01	0.001	0.0001
0.1	1.00	1.00	0.90	1.00	0.94	0.83	1.00	0.92	0.81
0.2	0.98	0.82	0.71	0.86	0.69	0.58	0.83	0.66	0.55
0.3	0.82	0.66	0.54	0.67	0.50	0.39	0.63	0.46	0.36
0.5	0.59	0.42	0.32	0.42	0.27	0.18	0.38	0.23	0.16

The agreement between the $r = 4.5$ result and that by the Canadian Foundation Engineering Manual (CFEM, 2006) is to some extent fortuitous, since the CFEM resistance factor is derived by calibration with past design methodologies which is quite different than the analytical approach taken here. The CFEM resistance factor apparently presumes a reasonable, but not significant, understanding of the soil properties under the footing (e.g. $r = 4.5$ rather than $r = 0$). The corroboration of the rigorous theory proposed here by an experience-based code provision is, however, very encouraging. The authors also note that the CFEM is the only source for which the live and dead load bias factors used in this study can be reasonably assumed to also apply.

Summary

The resistance factors recommended in Table 1 are conservative in (at least) the following ways; 1) it is unlikely that the correlation length of the residual random process at a site will equal the "worst case" correlation length, 2) the soil is assumed weightless in this study (adding weight increases bearing capacity), and 3) often more than one CPT is taken at the site in the footing region.

On the other hand, the resistance factors recommended in Table 1 are unconservative in (at least) the following ways; 1) measurement and model errors are not considered in this study. The statistics of measurement errors are very difficult to determine, since the true values need to be known. Similarly, model errors, which relate both the errors associated with translating measured values (e.g. CPT measurements to friction angle values) and the errors associated with predicting bearing capacity by an equation such as Eq. 3 with the actual bearing capacity are extremely difficult to measure simply because the true bearing capacity along with the true soil properties are rarely, if ever, known. In the authors' opinions this is the major source of unconservativism in the presented theory. When confidence in the measured soil properties or in the model used is low, the results presented here can still be employed by assuming that the soil samples were taken further away from the footing location than they actually were (e.g. if low-quality soil samples are taken directly under the footing, r = 0, the resistance factor corresponding to a larger value of r, say r = 4.5 m should be used), 2) the failure probabilities given by the above theory are slightly underpredicted when soil samples are taken at some distance from the footing. The effect of this underestimation on the recommended resistance factor has been shown to be relatively minor but nevertheless unconservative, and 3) c and ϕ are assumed independent, rather than negatively correlated, which leads to a somewhat higher probability of failure and correspondingly lower resistance factor, and so somewhat unconservative results. The authors note that this statement is contrary to the conclusion made in Fenton et al. 2003 (which was intended to refer to a positive correlation) – in any case, the effect of positive or negative correlation of c and ϕ was found in Fenton et al. to be quite minor.

To some extent the conservative and unconservative factors listed above cancel one another out. The comparison of resistance factors to other sources demonstrates that the 'worst case' theoretical results presented in Table 1 agrees quite well with current literature and LRFD code recommendations, assuming moderate variability and site understanding, suggesting that the theory is reasonably accurate. The theory provides an analytical basis to extend code provisions beyond calibration with the past.

One of the major advantages to a table such as 1 is that it provides geotechnical engineers with evidence that increased site investigation will lead to reduced construction costs and/or increased reliability. In other words, Table 1 is further evidence that you pay for a site investigation whether you have one or not (Institution of Civil Engineers, 1991).

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