NUMERICAL ANALYSIS OF BEAMS ON RANDOM ELASTIC FOUNDATIONS

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Abstract. The classical problem of a beam on an elastic foundation has long been of practical interest to geotechnical engineers, because it provides a framework for computing deflections, not only of horizontally oriented foundations, but also of vertically oriented laterally loaded piles. In both cases the soil is modeled as a system of elastic springs which can be calibrated to model the stiffness of the soils adjacent to the beam (or pile). In this study the influence of spatially random foundation stiffness on deformations of transversely loaded homogeneous beams is investigated. The approach involves a combination of finite element analysis, random field theory and Monte-Carlo simulations. The objective is to quantify the influence of statistically defined foundation stiffness (mean, standard deviation and spatial correlation length) on the mean and standard deviation of the beam deflection. This then leads naturally to a probabilistic interpretation in which, for example the probability of the top deflection of a laterally loaded pile exceeding some design threshold can be quantified.

Keywords: Piles, Probabilistic analysis, Random fields, Finite elements, Deflections

1. INTRODUCTION

Analysis of a beam on an elastic foundation is a classical problem first introduced by Winkler in the 19th century and later developed by many other investigators, most notably by Hetenyi [1]. In this paper we consider the response of a beam on a foundation with a spatially random stiffness. Similar studies have been performed using stochastic finite element methods (e.g. Ramu and Ganesar [2], Zhang and Ellingwood [3]) however in this study we will use the random finite element method (RFEM) first developed by Griffiths and Fenton

[4] and Fenton and Griffiths [5]. In this method, conventional finite element analysis of a beam on an elastic foundation (e.g. Smith and Griffiths [6]) is combined with random field generation (e.g. Fenton and Vanmarcke [7]) and Monte-Carlo simulations to develop output statistics of quantities such as the beam deflection. For example, in the analysis of a laterally loaded pile analysis in a spatially random soil, we might be interested in estimating the probability of the top deflection exceeding some allowable design value. This can be quantified by counting realizations that give excessive deflections or by fitting a probability density function to the output.

We start the paper by applying the finite element method to a simple analysis of a laterally loaded pile (e.g. Reese and Van Impe [8], Prakash and Sharma [9]) involving a soil of constant foundation stiffness which can be compared with the analytical solutions of [1]. The equation to be solved is

$$EI\frac{d^4y}{dx^4} + ky = q \tag{1}$$

EI = flexural rigidity of the beam, k = foundation stiffness, y = deflection and q = distributed loading.

Example Problem: A pile of length $L = 12.2$ m is driven into clay. The foundation stiffness is $k = 5774$ kPa and the pile stiffness is $EI = 9492$ kNm^2. The pile is subjected to a lateral top load of $P = 28$ kN. The characteristic length of the pile is given by λL, where $\lambda = \sqrt[4]{k/4EI}$. A "short beam" is given by $\lambda L \le \pi/4$, a "medium length beam" by $\pi/4 < \lambda L \le \pi$ and a "long beam" by $\lambda L > \pi$. The pile problem under consideration here has $\lambda L = 7.618$ so it may be considered "long".

The deflection at the top of the pile by Hetenyi's analytical solution is given by

$$\delta = \frac{2P\lambda}{k}\frac{\left(\sinh(\lambda L)\cosh(\lambda L) - \sin(\lambda L)\cos(\lambda L)\right)}{\sinh^2(\lambda L) - \sin^2(\lambda L)} = 0.00605 \text{ m} \tag{2}$$

The finite element solutions modeled with 2, 4 and 8 equal length elements are shown in Table 1. Very good agreement is obtained with the analytical solution, even using the coarsest dicretization. Figure 1 shows the computed deflected shape by finite elements (8 elements) compared with the solution from [1] over the whole length of the pile.

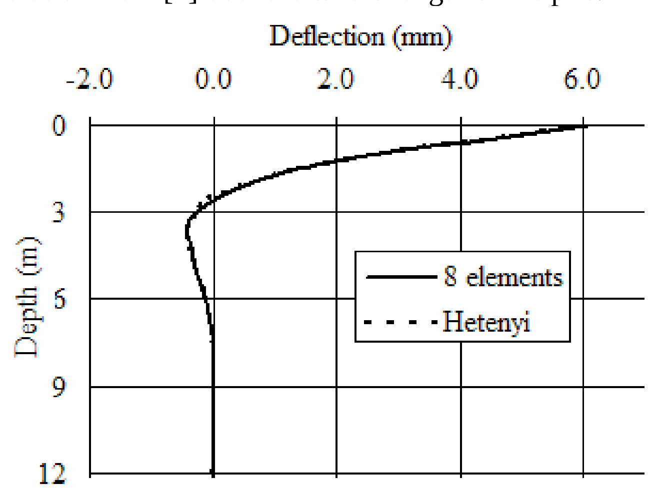

Figure 1. Deflected shape of pile from finite element and analytical solutions.

Table 1. Top deflection by finite element program and analytical solution (mm).

Hetenyi	FE solution, number of elements		
	2	4	8
6.05	5.82	5.92	6.04

2. RANDOM FIELDS

In this section we introduce spatially random soil adjacent to the pile. This is intended to model highly variable soil typical of some sites in which the soil stiffness is characterized by a mean, a standard deviation and a spatial correlation length. The Random Finite Element Method (RFEM), which combines finite element analysis with random field theory will be used in conjunction with Monte-Carlo simulations and has already been applied to several areas of geotechnical engineering by the authors [10]. The pile is divided into 100 elements and a random field of foundation stiffness is mapped onto the mesh taking full account of local averaging. Each element is assigned a k value which varies from one element to the next. The random field is defined by three parameters, the mean (μ_k), the standard deviation (σ_k) and the spatial correlation length $(\theta_{\ln k})$. A convenience dimensionless measure of the variability of data is given by the coefficient of variation, defined $V_k = \sigma_k / \mu_k$. The spatial correlation length is the distance over which the properties tend to be spatially correlated. A small spatial correlation length implies rapidly varying properties, while a large spatial correlation length implies gradually varying properties. Two random fields with the same mean and standard deviations could have quite different spatial correlation lengths. In the current work we have expressed the spatial correlation length in dimensionless form as

$$\Theta_{\ln k} = \frac{\theta_{\ln k}}{L} \tag{3}$$

It should be noted that in this study, the foundation stiffness k is assumed to be lognormally distributed, so the spatial correlation length is defined with respect to the underlying normal distribution of $\ln k$. Figure 2 shows a typical random field of foundation stiffness where dark and light regions depict, respectively, stiff and less stiff soil values.

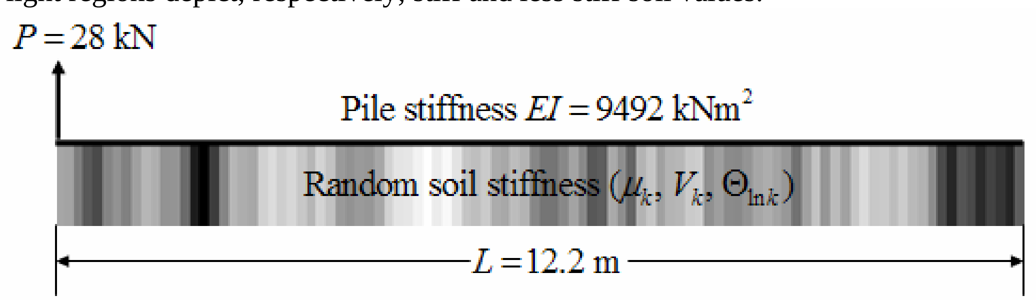

Figure 2. Typical random field of foundation stiffness mapped onto a mesh of 100 elements.

3. VARIANCE REDUCTION OVER A LINE FINITE ELEMENT

The RFEM takes full account of element size in the random field generation and the method delivers statistically consistent values of the locally averaged properties. It is assumed that the input statistics (μ_k, σ_k) are provided at the "point". For a line finite element of length

$\alpha\theta_{\ln k}$ and a Markov correlation function given by $\rho = e^{-2\tau/\theta}$ it can be shown (e.g. Vanmarcke [11]) that the local averaging variance reduction factor is given by

$$\gamma = \frac{2}{(\alpha\theta)^2}\int_0^{\alpha\theta}(\alpha\theta - x)e^{-2x/\theta}dx \tag{4}$$

where $\gamma = \sigma_{kA}^2 / \sigma_k^2$ and σ_{kA}^2 is the variance after local averaging. The variance reduction may be evaluated analytically from Eq. 4 to give

$$\gamma = \frac{1}{2}\frac{(2\alpha e^{2\alpha} - e^{2\alpha} + 1)e^{-2\alpha}}{\alpha^2} \tag{5}$$

It may be noted that local averaging affects both the standard deviation and the mean of the lognormal parameter.

4. RESULTS OF RFEM

The results of RFEM analyses with 5000 Monte-Carlo simulations are now presented, based on a range of parametric variations of, V_k and $\Theta_{\ln k}$. In all cases $\mu_k = 5774$ kPa.

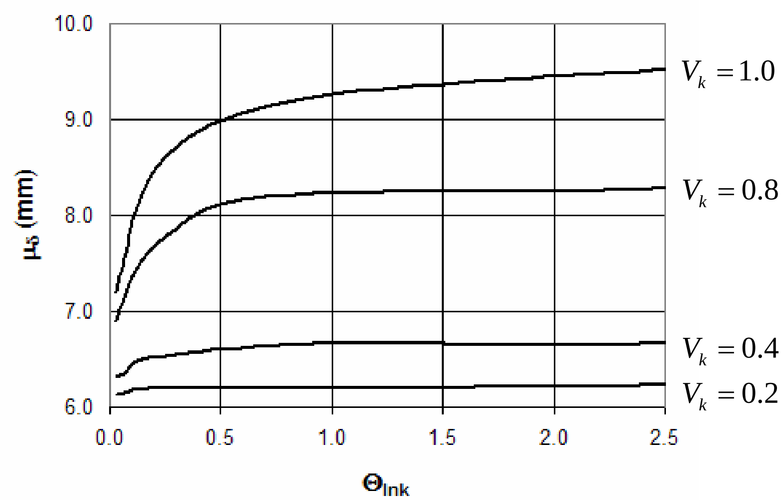

Figure 3. μ_δ vs. $\Theta_{\ln k}$ for different V_k values

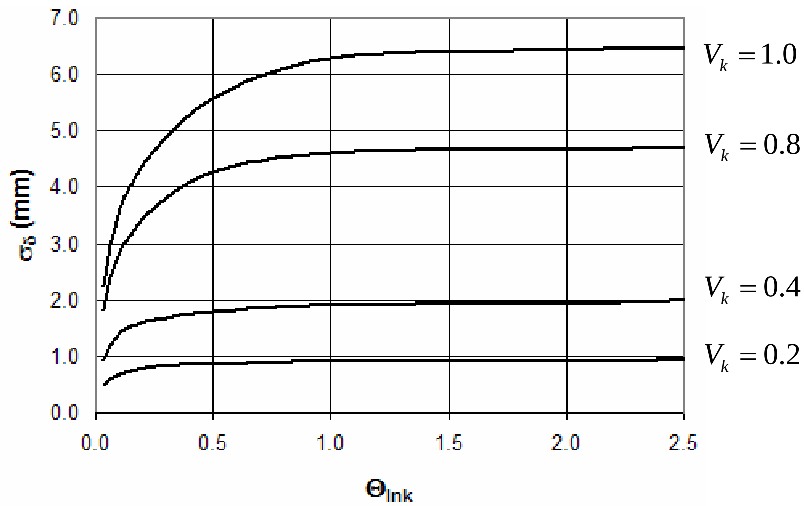

Figure 4. σ_δ **vs.** $\Theta_{\ln k}$ for different V_k values

It can be noted that both the mean and standard deviation of the top deflection increase with increasing values of V_k . They also increase with increasing $\Theta_{\ln k}$ for $\Theta_{\ln k} < 1$ but tend to remain essentially constant for $\Theta_{\ln k} > 1$

5. PROBABILISTIC INTERPRETATION

In order to make probabilistic interpretations from a Monte-Carlo analysis, we can either count the number of simulations that exceed the allowable deflection, or make an assumption about the probability density function (pdf) that best fits the output values. Since the foundation stiffness values were assumed to be lognormal, it seems reasonable to assume that the pdf of the top deflection is also lognormal. Figure 5 shows a histogram based on 5000 solutions from the RFEM runs compared with a smooth lognormal plot based on computed values of the top deflection δ given by μ_δ and σ_δ values. Although objective "goodness of fit" tests can be performed, in the interests of brevity we can note here that the lognormal fit seems reasonable.

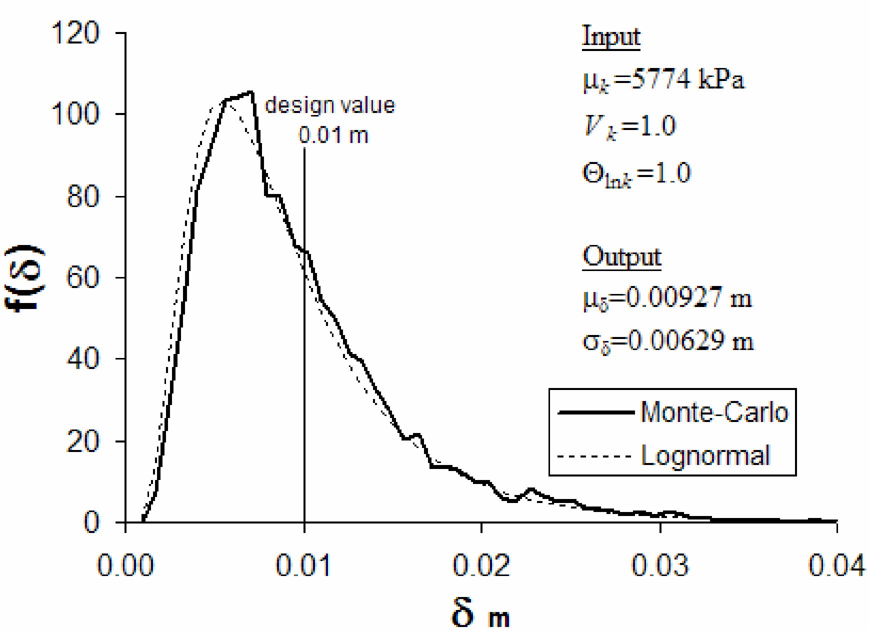

Figure 5. Histogram and lognormal fit for a typical set of computed deflection values.

The choice of a lognormal distribution of deflection makes the computation of probabilities easily obtained based on standard tables of the standard cumulative distribution function. For arguments sake, let us assume that the design has failed if the top deflection exceeds 10 mm. Thus for any particular parametric combination of V_k and $\Theta_{\ln k}$ we wish to estimate $\mathrm{P}[\delta > 10\mathrm{mm}]$.

Sample Calculation:

1) For input values $\mu_k = 5774$ kPa , $V_k = 1$ and $\Theta_{\ln k} = 1$
2) From Monte-Carlo simulations, $\mu_\delta = 9.27\,\mathrm{mm}, \sigma_\delta = 6.29\,\mathrm{mm}$ ($V_\delta = 0.679$ lognormal)

3) Obtain parameters of underlying normal distribution of $\ln\delta$,

$$\sigma_{\ln\delta} = \sqrt{\ln(1+V_\delta^2)} = 0.62 \text{ and } \mu_{\ln\delta} = \ln\mu_\delta - \frac{1}{2}\ln\left\{1+V_\delta^{\,2}\right\} = 2.04$$

4) $$\mathrm{P}\left[\delta > 10\text{mm}\right] = 1-\Phi\left[\frac{\ln 10 - \mu_{\ln\delta}}{\sigma_{\ln\delta}}\right] = 1-\Phi\left[0.43\right] = 0.333\ (33.3\%)$$

where $\Phi(.)$ is the standard cumulative distribution function.

Figure 6 shows results of similar probabilistic calculations for other values of V_k and $\Theta_{\ln k}$. Probabilities corresponding to $V_k = 0.2$ were negligible so they are not shown.

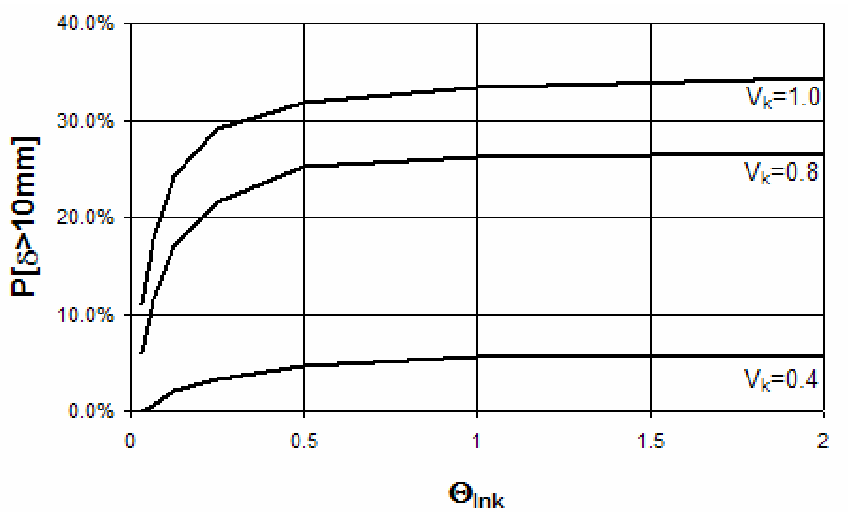

Figure 6. $\mathrm{P}\left[\delta > 10\text{mm}\right]$ vs. $\Theta_{\ln k}$ for different V_k values

6. CONCLUSIONS

The paper has described analysis of a laterally loaded pile embedded in a soil with statistically defined stiffness. The pile was modeled as a "beam on an elastic foundation" using the Random Finite Element Method (RFEM) with the soil stiffness defined by its mean, standard deviation and spatial correlation length. Monte-Carlo simulations led to stable output statistics of the pile top deflection, from which probabilistic conclusions were reached. A full range of parametric solutions were considered involving the spatial correlation length and the coefficient of variation of the soil stiffness. The results show that increasing the coefficient of variation of the foundation stiffness (with the mean held constant) results in a significant increase in mean and standard deviation of pile top deflection. Increasing the spatial correlation length of the foundation stiffness also results in increased mean and standard deviation of top deflection, although this effect levels out when the spatial correlation length approaches the length of the pile itself. When these results were interpreted probabilistically, a similar trend was observed. For the test problem, the probability of "design failure", defined as excessive pile top deflection, was as high as 30% for highly variable soil. The program developed as part of this study has much potential for further investigations of the response of piles in random soils. An important ongoing refinement is to recognize increased stiffness with depth within a random framework.

7. REFERENCES

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