Effect of Slope Height and Gradient on Failure Probability

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ABSTRACT

In the development of a regional landslide hazard measure, one must take into account the various slopes occurring in the region under consideration. Regions of large areal extent may contain hundreds or even thousands of discrete slopes, with different heights and gradients, any of which may potentially fail. If the regional landslide hazard is defined as the probability that at least one slope, of some size, will fail within the region, then the regional hazard is a function of several parameters: the number of slopes in the region, and the composition, height, and gradient of each slope. Digital elevation models can be used to estimate the distribution of slope gradients and heights as a function of digital resolution. What is further needed to assess the regional hazard is the probability of slope failure as a function of slope gradient and height for various soil compositions.

This paper develops conditional failure probabilities of slopes for various slope angles, heights, and coefficients of variation of soil strength. The paper focuses on one type of slope -- purely cohesive on a bedrock foundation -- to describe and illustrate the more general methodology which would be applicable to any region. The paper demonstrates how the results can be used, in combination with digital elevation model distributions, to obtain regional landslide hazard measures leading to regional landslide risk assessments.

INTRODUCTION

This paper investigates the probability of slope failure as a function of the random ground parameters making up the slope, and the height and gradient of the slope. The results are an essential first step in the development of regional landslide hazard measures, and so some background into how the results fit into estimates of landslide hazard will first be discussed. Note, however, that this paper does *not* estimate regional landslide hazard levels.

With the availability of digital elevation models at increasingly fine resolutions, the ability to look at the variety of slopes occurring in a spatial region improves. In turn, landslide hazard assessments over spatial regions are increasingly in demand. There are alternative possible definitions for regional landslide hazard, but they all generally involve the probability of slope failure. One possible definition of regional landslide hazard is the probability that at least one slope in the region will fail during some specified period of time, *T*. This definition was adopted by Fenton et al. (2013). Estimating this probability requires that the total number of slopes in the region be known. A similar, but simpler, measure of regional landslide hazard is the probability, p_f, that a randomly selected slope in the region will fail during time *T*. The probability that a randomly selected slope in a region will fail requires at least the following information;

1. the relative frequencies (occurrence probability) of the types of ground making up the slopes in the region and the statistics of the worst ground shear strength parameters (e.g., mean and variance of the friction angle in ground type 1, etc) likely to be present over the design period, *T*,
2. the relative frequencies (distributions) of slope heights and gradients occurring in the region.

If this information is known, then the probability that a randomly selected slope in a region will fail can be computed using the total probability theorem as

$$p_f = \sum_S \sum_\beta \sum_H \mathrm{P}\left[F \mid S \cap \beta \cap H\right] \mathrm{P}\left[\beta \cap H\right] \mathrm{P}\left[S\right] \tag{1}$$

where *F* is the event that the slope fails, *S* is the type of ground, β is the angle the slope makes with the horizontal, and *H* is the slope height. The fraction of slopes having ground type *S*, P[*S*], is obtained by investigation, as are the statistics of their shear strength parameters. The probability that a randomly selected slope has height *H* and slope angle β, $\mathrm{P}\left[\beta \cap H\right]$ can be obtained by a statistical analysis of digital elevation model (DEM) data.

The only component on the right hand side of Eq. 1 that remains to be determined is the probability of slope failure given the ground type and its statistics, *S*, the slope height, *H*, and the slope angle, β. The goal of this paper is to present preliminary results relating to this conditional probability, $\mathrm{P}\left[F \mid S \cap \beta \cap H\right]$. The study is preliminary in the sense that only one type of soil is considered, *S* = a purely cohesive soil, and only slopes founded on bedrock are considered.

THE RANDOM FINITE ELEMENT METHOD

As mentioned above, only purely cohesive soils are considered in this study – the cohesion, c_u, is assumed to be spatially varying with a marginal lognormal distribution, so that ln c_u is normally distributed. The random field used to model c_u has three parameters; its mean, μ_{c_u}, its standard deviation, σ_{c_u} and its spatial correlation length, $\theta_{\ln c_u}$. The lognormal parameters, $\mu_{\ln c_u}$ and $\sigma_{\ln c_u}$ are obtained through the transformations

$$\sigma^2_{\ln c_u} = \ln(1+\upsilon^2_{c_u}), \quad \mu_{\ln c_u} = \ln(\mu_{c_u}) - 0.5\sigma^2_{\ln c_u} \tag{2}$$

where $\upsilon_{c_u} = \sigma_{c_u} / \mu_{c_u}$ is the coefficient of variation of c_u.

For generality, the soil shear strength is expressed in dimensionless form $C = c_u / (\gamma H)$, where γ is the soil's unit weight and H is the slope height. If the cohesion field is spatially uniform, then the slope will fail if $C < N_s$, where N_s is Taylor's stability number (Taylor, 1937 and 1948, and Baker, 2003). Without loss of generality, μ_{c_u} and γ are both taken to equal 1.0, and slope heights H near failure are investigated. That is, a range of borderline slope heights H are investigated having probabilities of slope failure typically somewhere between 0 and 1 (rather than at 0 or 1). The failure probability results presented in this paper can be used for real problems simply by comparing the statistics of the actual $C = c_u / (\gamma H)$, i.e. the mean, standard deviation, and correlation length of C, to values considered in this study. The parameters γ and H are taken to be deterministic constants, and so if c_u is lognormally distributed, as assumed, then C is also lognormally distributed with parameters,

$$\mu_{\ln C} = \mu_{\ln c_u} / \ln(\gamma H), \quad \sigma_{\ln C} = \sigma_{\ln c_u} \tag{3}$$

with the same correlation length, i.e. $\theta_{\ln C} = \theta_{\ln c_u}$.

The correlation length, $\theta_{\ln c_u}$ is the distance beyond which values of $\ln c_u$ are largely independent – small values of $\theta_{\ln c_u}$ lead to fields which vary rapidly in space, whereas large values of $\theta_{\ln c_u}$ lead to smoother, more slowly varying fields. The correlation length is measured with respect to $\ln c_u$ since $\ln c_u$ is normally distributed and the joint normal distribution is simply specified by its mean, variance, and correlation length. The actual correlation coefficient between values of $\ln c_u$ is assumed to be given by the Markov correlation function,

$$\rho\left(\underset{\sim}{\tau}\right) = \exp\left\{-\frac{2\left|\underset{\sim}{\tau}\right|}{\theta_{\ln c_u}}\right\} \tag{4}$$

where $\underset{\sim}{\tau}$ is the lag vector between $\ln c_u$ at two points and $\left|\underset{\sim}{\tau}\right|$ is its Euclidean length. The correlation length can also be non-dimensionalized by refering to $\Theta = \theta_{\ln C} / H$. In general, a slope having the same statistics of C and same Θ value as one of the cases studied herein will have the same failure probability.

The goal of this paper is to determine the probability of slope failure, p_f, as a function of four quantities;

1) the slope height, H,
2) the slope angle, β,
3) the cohesion coefficient of variation, υ_c, and
4) the relative correlation length, Θ.

Figure 1 illustrates two possible realizations of slopes having identical (initial) geometry. The distribution of soil strengths in the upper slope was such that the slope did not fail, whereas paths through the soil in the lower slope were found which were sufficiently weak to lead to slope failure. Due to spatial variability, the failure mechanism is seen in the lower slope to be much more complicated (and realistic) than assumed in classical slope stability solutions.

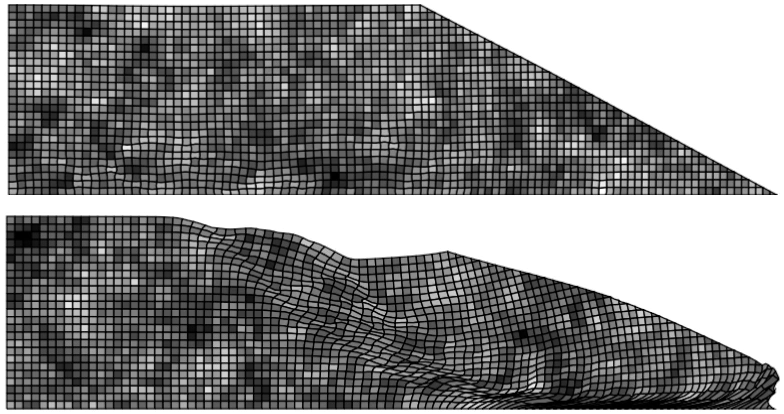

Figure 1. Two realizations of a slope with height *H* = 8 m and slope angle $\beta = 30^\circ$.

If the correlation length Θ is infinite, then all cohesion fields are *uniform*, i.e. random from realization to realization but spatially constant within each realization. That is, *C* becomes a single random variable. In this case, the probability of failure, p_f, can be obtained as

$$p_f = \mathrm{P}\left[C < N_s\right] = \Phi\left(\frac{\ln N_s - \mu_{\ln C}}{\sigma_{\ln C}}\right) \tag{5}$$

where N_s is Taylor's stability numbers (see, e.g., Baker, 2003) and $\Phi(\cdot)$ is the standard normal cumulative distribution function. When the correlation length is less than infinity, *C* becomes a spatially variable random field and Eq. 5 can no longer be used as is: *C* must be replaced by some equivalent random variable whose distribution is a function of the slope geometry and the mean, variance, and correlation length of *C*. The determination of this equivalent random variable is a complex task, which will not be attempted in this preliminary study. Instead the failure probabilities, p_f, will be estimated using Monte Carlo simulation – see Figure 1 for an example of two possible realizations.

The Random Finite Element Method proceeds by combining a random field model for the cohesion field (as discussed above) with a finite element discretization of the slope. Repeated finite element analyses of a sequence of n_{sim} realizations and counting the number which fail, n_f, allows the estimation of the failure probability, $\hat{p}_f = n_f / n_{sim}$. The standard error of this estimate is $\sigma_{\hat{p}_f} \simeq \sqrt{\hat{p}_f(1-\hat{p}_f)/n}$.

The four parameters to be varied in the study are *H*, β, υ_c, and Θ. In general, at least one further geometrical parameter would be the depth of a soil foundation underlying the slope. To restrict the number of parameters being varied to the four mentioned above, the foundation is assumed to be bedrock, so that the failure mechanism must remain within the slope. Work on the more general case where the slope has an underlying foundation is ongoing (see, e.g., Huang et al., 2010). The values used for the four parameters in the Monte Carlo study are as follows;

- slope angle $\beta = 10°$, 20°, 30°, 40°, 50°, and 60°. These were selected arbitrarily to represent a reasonable range. The stability numbers corresponding to these slopes are N_s= 0.081, 0.109, 0.136, 0.159, 0.174, and 0.191.

- slope height, H = 2, 4, 6, and 8 m. These heights were selected so that $\mu_{\ln C} = \mu_{\ln c_u} / \ln(\gamma H) = 1/H$ in this study approximately spanned the N_s range, with emphasis on the steeper slopes. In other words, for H ranging from 2 to 8 m, the range in μ_C is from 0.125 to 0.5, i.e. from below the critical value for the 30° slope to above the critical value for the 60° slope. Future studies will expand this range.
- cohesion coefficient of variation, υ_c = 0.1, 0.2, 0.3, 0.4, and 0.5. These values range from well below typical values stated in the literature (e.g. Phoon and Kulhawy, 1999) to well above.
- relative correlation length, $\Theta =$ 0.1, 0.2, 0.3, 0.5, 0.7, 1.0, 1.2, 1.5, 2.0, 5.0, 10.0, and 50.0. In other words the actual correlation length ranged from 1/10 of the slope height to 50 times the slope height.

For each parameter set, n_{sim}= 2000 realizations were performed so that the standard error on the failure probability estimates is $0.022\sqrt{\hat{p}_f(1-\hat{p}_f)}$. The total number of parameter sets considered was 6×4× 5×12 = 1440 and the entire study took about 2000 CPU-hours on modern (2012) CPUs.

RESULTS AND DISCUSSION

The probability of failure is a function of four parameters in this study, and so there are many ways of looking at the simulation results. Due to space constraints, not all of the results are presented in this paper. Rather, some representative results are shown to illustrate how the failure probability is affected by the four parameters. The paper restricts its attention to simulation results – work is ongoing on a more general prediction model, which is a function of these four variables and which summarizes the results in a compact mathematical expression or probabilistic model.

Figure 2 illustrates how p_f varies with correlation length for a specific slope geometry (H = 6 m and $\beta = 40°$, for which N_s = 0.159); this geometry was selected here because its behaviour at small correlation lengths shows a bifurcation, tending towards 1.0 when $\upsilon_c \geq 0.3$ and towards 0.0 otherwise. To explain why this is so requires an understanding of what is happening at the element level as $\Theta \to 0$ (which for H = 6 m being fixed means that $\theta_{\ln c_u} \to 0$. Since c_u (or C) is assumed to be lognormally distributed, the simulation proceeds by generating a normally distributed random field which gives local averages over each element, G_i, for the i^{th} element (via the Local Average Subdivision method, Fenton and Vanmarcke, 1990). The lognormally distributed final cohesion field is then computed as $\exp\left\{\mu_{\ln c_u} + \sigma_{\ln c_u} G_i\right\}$ for each element i = 1, 2, ..., n_{el},where n_{el} is the number of elements. This transformation means that the final lognormally distributed cohesion field is made up of a series of *geometric* averages of the underlying normally distributed field over each element. The geometric average, discussed in detail by Fenton and Griffiths (2008), has two limits; the (non-random) median of c_u when $\theta_{\ln c_u} \to 0$ and a random variable having the same marginal distribution as c_u when $\theta_{\ln c_u} \to \infty$. The median of c_u is $\exp\left\{\mu_{\ln c_u}\right\} = 1/\sqrt{1+\upsilon_c^2}$. Thus, when $\theta_{\ln c_u} \to 0$

the non-dimensional C becomes deterministic and equal to 0.1658, 0.1634, 0.1596, 0.1548, and 0.1491 for υ_c = 0.1, 0.2, 0.3, 0.4, and 0.5, respectively. In other words, when $\upsilon_c > 0.3$, C is less than N_s = 0.159, in which case the failure probability tends to 1.0. When $\upsilon_c < 0.3$, C is greater than N_s = 0.159 so that the failure probability becomes 0 as $\theta_{\ln c_u} \to 0$. Note that the case where $\upsilon_c = 0.3$ is borderline – the finite element analysis is suggesting that the corresponding $C = 0.1596$ is actually less than N_s for this slope. The authors note that the N_s used here is obtained from curves presented in the literature, and so small discrepancies between finite element results and published stability numbers are to be expected.

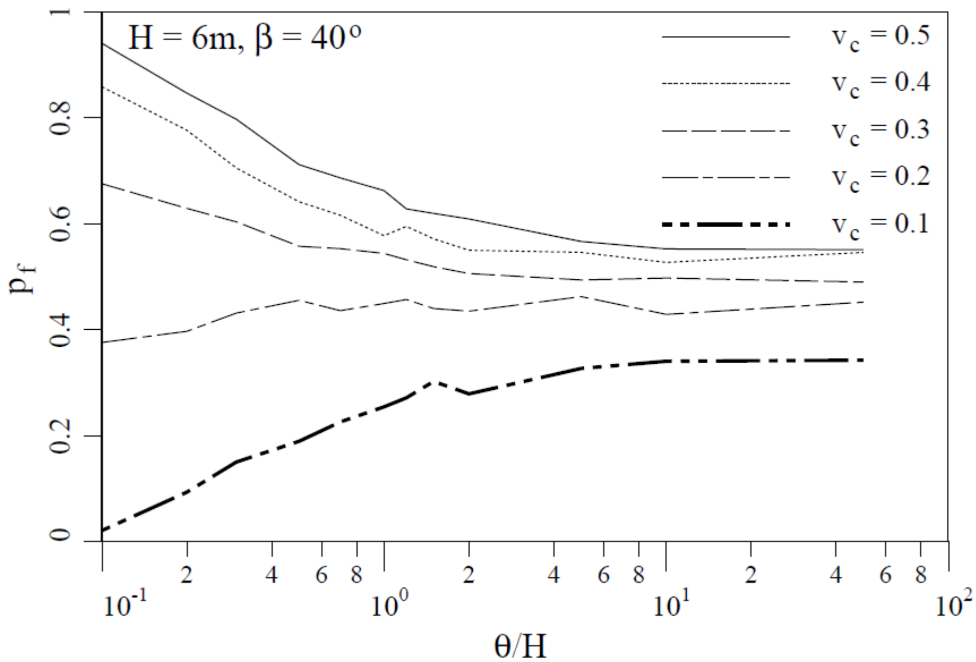

Figure 2. Slope failure probability versus $\theta_{\ln C}/H$ for a slope having height *H* = 6 m, $\beta = 40°$ and various cohesion coefficients of variation, υ_c.

At the opposite end of Figure 2, as $\Theta \to \infty$, the failure probability is tending to that obtained by the single random variable result given by Eq. 5. The largest discrepancy between the simulation based result at $\Theta = 50$, $\hat{p}_f = 0.5465$, and that predicted by Eq. 5, $\hat{p}_f = 0.5281$, occurs when $\upsilon_c = 0.4$. This relative difference of about 3% is to be expected, even if due to sampling errors alone.

Figure 3 illustrates how p_f varies with slope height, *H*, for a slope angle of $\beta = 50°$, $\upsilon_c = 0.3$ and various correlation lengths. Not surprisingly, the failure probability increases as the slope height increases for a fixed slope angle. The curve for $\Theta = \theta / H = \infty$ is given by Eq. 5 and can be seen to be quite close to the $\Theta = 10$ curve, as expected. When $\Theta = 0$, every element takes on the value of the median of c_u, which is $\exp\left\{\mu_{\ln c_u}\right\} = 0.9578$ for $\upsilon_c = 0.3$ as assumed in Figure 3 (with $\mu_{c_u} = 1.0$). In this case, C

is deterministic and has value 0.9578/$0.9578/(\gamma H) = 0.9578/H$. The value of H when $C = N_s$ is found by solving 0.9578/H = 0.174 which gives H = 5.50 m. In other words, when $\Theta = 0$, slopes having height in excess of 5.50 m fail and slope having height less than 5.50 m survive. The curve corresponding to $\Theta = 0$ is the vertical dashed line at H = 5.50. Similarly, the theoretical curve corresponding to the mean of c_u, which is $\mu_{c_u} = 1.0$ is found by solving 1/H = 0.174 which gives H = 5.747 which is also shown in Figure 3.

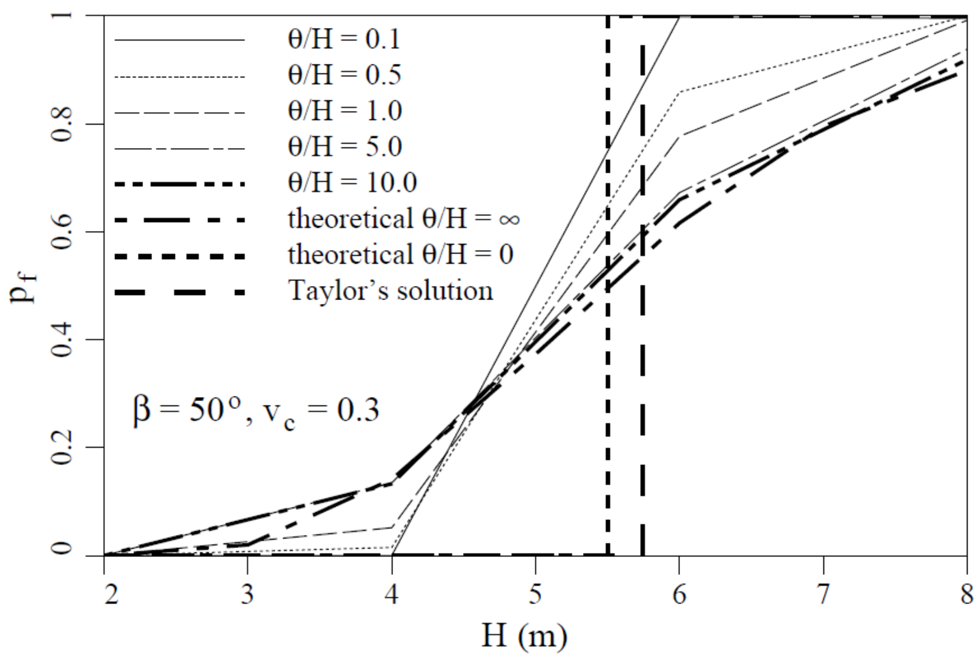

Figure 3. Slope failure probability versus slope height, *H* for a slope having angle $\beta = 50°$ and coefficient of variation of $\upsilon_c = 0.3$.

Interestingly enough, there appears to be a cross-over point at about H = 4.5 m above which p_f becomes lower for higher correlation length (see also Griffiths and Fenton, 2000 and 2004). The authors note, however, that the failure probabilities are only known for H = 2, 4, 6, and 8 m, so the most that can be really said about the cross-over point is that it occurs somewhere between 4 and 6 m. Slopes having heights less than this cross-over point have increasing failure probability with increasing correlation length. This phenomenon is linked to the relationship between the equivalent value of c_u and the stability number of the slope. The equivalent value of c_u would be that spatially uniform value which has the same factor of safety as provided by the spatially varying C. A precise determination of the value of H at the cross-over point should allow the estimation of the mean equivalent c_u (equal to $\gamma H N_s$). It is suspected that this cross-over point will lie to the left of the $\Theta = 0$ curve. Work is ongoing to increase the resolution in H values used in the simulation to determine this point, but this work was not complete at the time of writing.

Figure 4 shows essentially the same information as Figure 3 except for a shallower slope, $\beta = 30°$. As expected, the shallower slope allows a greater slope height before the failure probability starts to increase. In this particular case, the cross-over point seems to be only slightly below the height corresponding to the median ($\Theta = 0$).

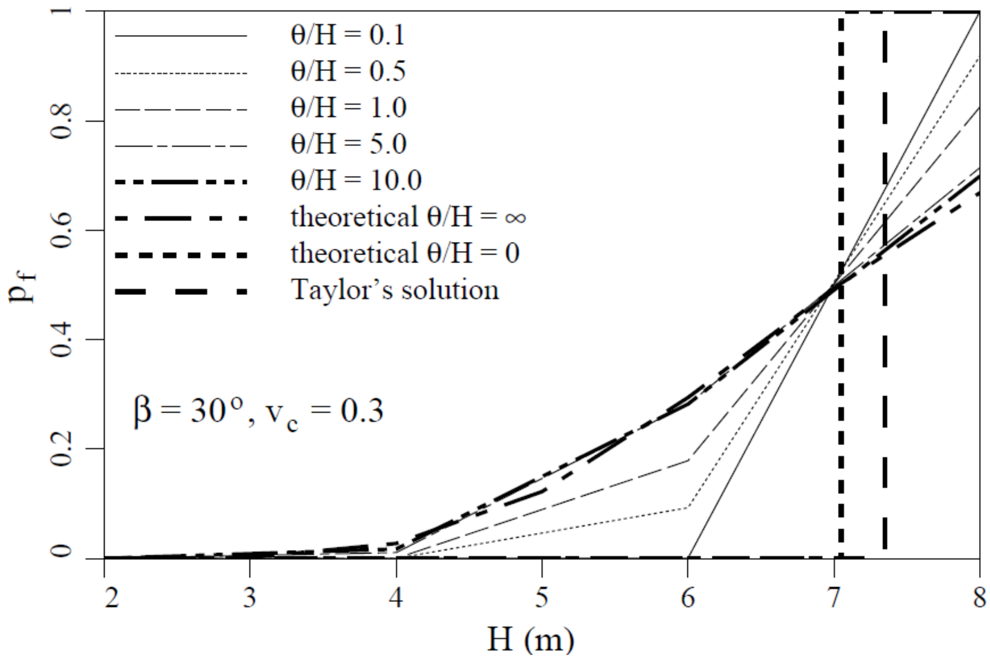

Figure 4. Slope failure probability versus slope height, *H* for a slope having angle $\beta = 30°$ and coefficient of variation of $\upsilon_c = 0.3$.

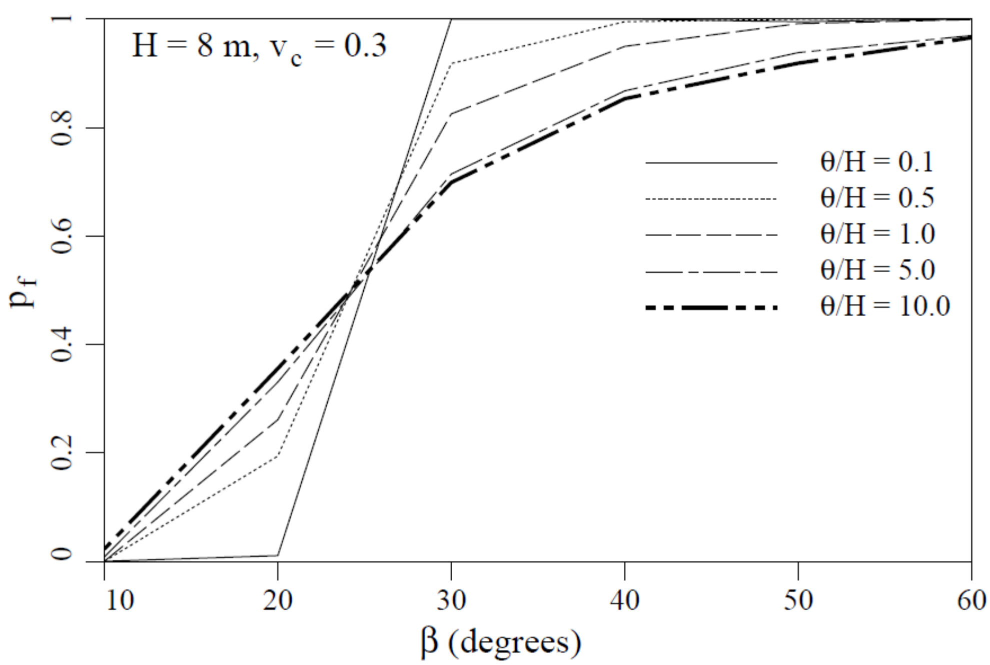

Figure 5. Slope failure probability versus slope angle, β for a slope having height *H* = 8 m and coefficient of variation of $\upsilon_c = 0.3$.

Figure 5 illustrates how the probability of slope failure is influenced by the slope angle for a fixed slope height (H = 8 m) and coefficient of variation ($\upsilon_c = 0.3$). The behaviour is very similar to that seen in Figures 3 and 4. In fact, Figures 3 and 4 are comparing how the equivalent value of $c_u/(\gamma H)$ compares to a fixed stability number, N_s, while Figure 5 is comparing how a fixed equivalent $c_u/(\gamma H)$ compares to a changing stability number (changing with slope angle). So again, there exists a cross-over point where slope angles in excess of the point have decreasing failure probabilities with increasing correlation length.

Figure 6 shows how the slope failure probability varies with the coefficient of variation of the cohesion field for the specific case where H = 8 m and $\beta = 20°$. This case was selected because it also displays a cross-over point at about $\upsilon_c = 0.46$. It is known that the value of υ_c affects the median of c_u, since the median is equal to $\exp\{\mu_{\ln c_u}\}$ which, for the $\mu_{c_u} = 1$ assumed here, is equal to $1/\sqrt{1+\upsilon_c^2}$.

What Figure 6 is demonstrating is that when υ_c is greater than about 0.46, the failure probability starts to decrease with increasing correlation length (for fixed H). Conversely, when υ_c is less than about 0.46, the failure probability increases with increasing correlation length. The significance of this cross-over point is probably similar to that seen in previous figures – in other words, the value of υ_c affects the equivalent value of C relative to the stability number N_s, as does the slope height (in Figures 3 and 4) and slope angle (in Figure 5).

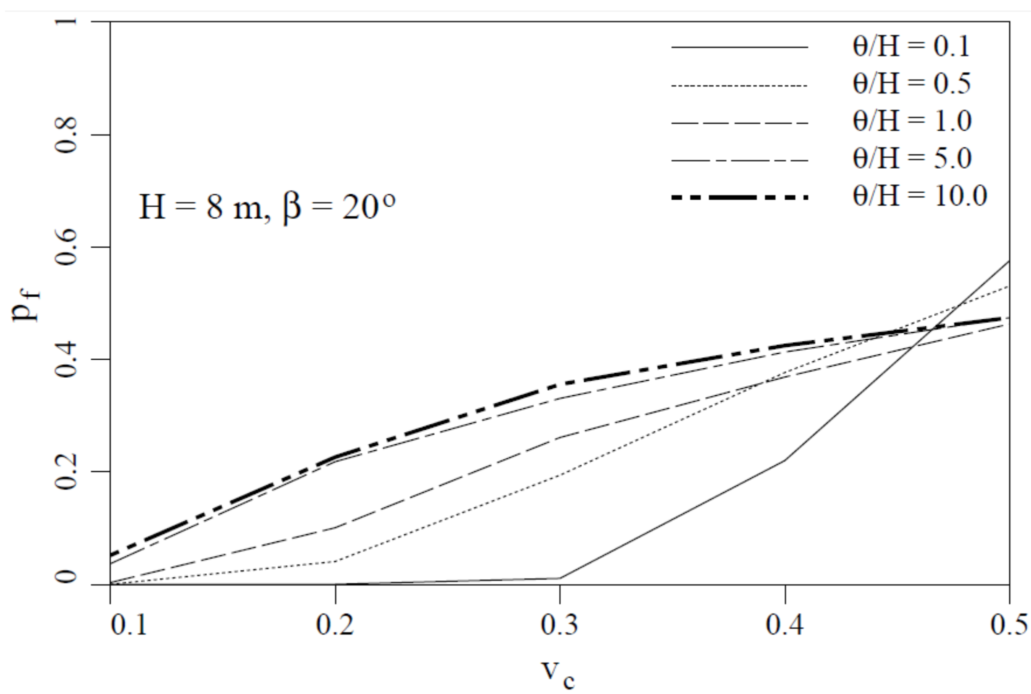

Figure 6. Slope failure probability versus cohesion coefficient of variation, υ_c for a slope having height H = 8 m and angle $\beta = 20°$.

CONCLUSIONS

The paper presents results from an ongoing study into the relationship between slope failure probability and the height, angle, and soil strength statistics of the slope. Some limitations of the study are as follows;

1) only cohesive soils are considered. The extension to more general $c-\phi$ soils is trivial, and follows exactly the same methodology, but has yet to be done.
2) slopes having a foundation layer which could also fail have not been considered. Again, this is a relatively simple extension, although it significantly increases the number of parameter sets that need to be considered.

The current study, i.e. restricted as mentioned above, is valuable in the sense that it allows the basic probabilistic behaviour to be more easily investigated – results from this study will no doubt aid in deciphering a more general study.

The main observation derived from this study is that the general probabilistic behaviour, in particular whether the probability of failure decreases or increases with increasing correlation length depends on whether the equivalent value of $C = c_u / (\gamma H)$ is above or below the stability number for the slope. This statement probably also holds even for slopes having a foundation layer. The equivalent value of C is unknown, since c_u is a random field, but it is possible that its statistics can be obtained by finding the cross-over points seen in Figures 3 through 5. In turn this may lead to a simple analytical prediction for the failure probability of a slope as a function of the slope's height, angle, and soil strength statistics and is a topic under continued investigation.

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