Reliability-Based Geotechnical Design: Towards a Unified Theory

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ABSTRACT: A review of theoretical probabilistic models devised over the last decade for geotechnical reliability-based design reveals that they generally follow the same form. That form can be used to develop a unified reliability-based design approach that includes the effects of spatial variability in the ground, site understanding, and the severity of failure consequences. This paper develops and describes the resulting unified model, along with recommendations regarding its use in practice. The approach can be used to directly provide required resistance factors for use in an LRFD format.

1. INTRODUCTION

There is a real desire in the geotechnical community to account for site understanding in the process of achieving economical, yet safe geotechnical designs. To accomplish this, it makes sense to have a resistance factor which is adjusted as a function of site understanding and that allows maintaining overall safety at a common target maximum failure probability as well as to demonstrate the direct economic advantage of increased site understanding.

The overall safety level of any design should depend on at least three factors: 1) the uncertainty in the loads, 2) the uncertainty in the resistance, and 3) the severity of the failure consequences. In most modern codes, these three items are assumed independent of one another and are thus treated separately. The load factors handle the uncertainties in the loads and, on the load side, failure consequences are handled by applying an importance factor to the more uncertain and site specific loads (e.g. earthquake, snow, and wind). Uncertainties in resistance are handled by resistance factors that are usually specific to the material used in the design. When dealing with a highly variable and site specific material such as the ground, it makes sense to apply a factor that depends on both the resistance uncertainty and on the consequences of failure. Similar to the multiplicative approach taken in structural engineering, where the overall load factor is a product of a load factor and an importance factor, the overall resistance factor applied to geotechnical resistance is taken here to consist of two parts which are multiplied together;

1. a resistance factor, φ_{gu} or φ_{gs}, which accounts for resistance uncertainty. This factor aims to achieve a target maximum acceptable failure probability equal to that used for geotechnical designs for typical failure consequences.
2. a consequence factor, Ψ, which accounts for failure consequences. Essentially, $\Psi > 1$ if failure consequences are low and $\Psi < 1$ if

failure consequence exceed those of typical geotechnical systems. The basic idea of the consequence factor is to adjust the maximum acceptable failure probability of the design down for high failure consequences, or up for low failure consequences.

This paper will consider limit state design (LSD) of geotechnical systems within a load and resistance factor design (LRFD) framework. The goal is to provide a single theoretical model which can be used to determine the resistance and consequence factors required to achieve a target maximum acceptable failure probability for a variety of geotechnical design problems.

Within the LRFD framework, geotechnical designs proceed by adjusting the resistance parameters (usually the foundation geometry) so that the factored geotechnical resistance at least equals the effect of factored loads. For example, for ultimate limit states (ULS), this means that the geotechnical design should satisfy an equation of the form

$$\Psi_u \varphi_{gu} \hat{R}_u \geq \sum I_i \alpha_i \hat{F}_i \qquad (1)$$

in which Ψ_u is a consequence factor, φ_{gu} is the geotechnical resistance factor, and $\hat{R}_u$ is the characteristic ultimate resistance, all at the ULS. The right-hand-side consists of I_i, an importance factor, multiplying the i^{th} factored load effect, $\alpha_i \hat{F}_i$. A similar equation must be satisfied for serviceability limit states (SLS), with the subscript *u* replaced by *s*.

Since the focus of this work is on calibrating resistance and consequence factors, which are applied to the characteristic resistance, the importance factors, I_i, will be assumed to have values 1.0. In addition, only dead and live loads will be considered in this study. If the characteristic total load, $\hat{F}_T$, is defined as the sum of the factored characteristic loads,

$$\begin{aligned} \hat{F}_T &= \alpha_L \hat{F}_L + \alpha_D \hat{F}_D \quad \text{for ULS design} \\ &= \hat{F}_L + \hat{F}_D \qquad\quad\;\; \text{for SLS design} \end{aligned} \qquad (2)$$

where it is assumed that the SLS load factors are 1.0, then the LRFD eq. (1) simplifies to

$$\begin{aligned} \Psi_u \varphi_{gu} \hat{R}_u &\geq \hat{F}_T \\ \Psi_s \varphi_{gs} \hat{R}_s &\geq \hat{F}_T \end{aligned} \qquad (3)$$

for the ULS and SLS cases, respectively. Three failure consequence levels will be considered in this paper;

1) *high consequence*: failure of the supported structure has large safety and/or financial consequences (e.g., hospitals, schools, and lifeline highway bridges),
2) *typical consequence*: has failure consequences typical of the majority of civil engineering projects, and
3) *low consequence*: failure of the supported structure has little or no safety and/or financial consequences (e.g., low use storage facilities or low use bridges).

Most designs will be aimed at the typical failure consequence level.

2. THEORETICAL FAILURE PROBABILITY AND DERIVED DESIGN FACTORS

The theoretical framework required to estimate the failure probability of a geotechnical system should consider;

1) uncertainty in the loads, and
2) uncertainty in the resistance, including random field models of the ground to characterize its natural spatial variability, along with prediction model uncertainty, and uncertainty in ground strength parameters (due to measurement errors and lack of sufficient sampling) within the zone of influence under and around the foundation being designed.

In its simplest form, a geotechnical system fails if its actual resistance, *R*, is less than the supported total load, F_T, any time during the system's design life. For example, Figure 1 illustrates a bearing failure mechanism which might occur at an instant in time during the design life of a footing. Rather than the traditionally assumed symmetric double log-spiral failure mechanism predicted when the ground properties are spatially constant, the failure mechanism that occurs when ground properties vary spatially follows the weakest path, resulting in non-symmetric and sometimes quite erratic failure paths (Fenton and Griffiths, 2008).

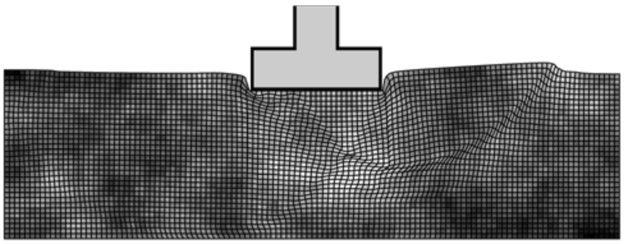

Figure 1: Bearing failure of a shallow foundation on a spatially variable soil.

The major challenge in reliability-based design is how to capture the weakest path behaviour of the ground in a way that is simple enough to use in practice. The key to answering this question is to replace the spatial variability of the ground by a single random variable which yields the same probabilistic behaviour as the spatially variable ground. In other words, is there a single random variable which gives the same failure probability as does the random field? Clearly, there must be. Consider, for example, the settlement of a shallow foundation where performance failure is defined as the event that the actual foundation settlement, δ, exceeds the serviceability limit, δ_{max}, i.e.,

$$p_f = \mathrm{P}\left[\delta > \delta_{max}\right] \tag{4}$$

The actual settlement, δ, is a function of the random loads the foundation sustains over time, the foundation geometry, and the random (usually non-linear and time varying) compressibility field of the ground under the footing. Thus, δ is a very complicated function of many random variables. Nevertheless, δ is a single random variable which has some distribution. If that distribution can be found, then p_f can be determined.

To illustrate the process in a geotechnical context, consider the bearing failure of a strip footing supported by a $c-\phi$ soil, as shown in Figure 1 (following Fenton et al., 2008). To simplify the illustration, the soil will be considered weightless with no foundation embedment nor surcharge.

The characteristic resistance becomes

$$\hat{R}_u = B\hat{c}\hat{N}_c \tag{5}$$

where B is the footing width, $\hat{c}$ is the characteristic cohesion, and the characteristic bearing capacity factor, $\hat{N}_c$, is given by (see e.g., Prandtl,1921, and Griffiths et al., 2002)

$$\hat{N}_c = \frac{\exp\{\pi \tan\hat{\phi}\}\left(\tan\hat{\phi} + \sqrt{1+\tan^2\hat{\phi}}\right)^2 - 1}{\tan\hat{\phi}} \tag{6}$$

The characteristic ground parameters (e.g., cohesion and friction angle) are obtained through a site exploration program. Although the definition of 'characteristic' varies quite widely around the world, it is assumed here that the characteristic values are 'a cautious estimate of the mean ground parameter'. They will be taken to be some sort of average, usually a geometric average since it is low-strength dominated, of the soil sample.

Using eq. (5) in eq. (3), the LRFD equation at ULS becomes

$$\Psi_u \varphi_{gu} B\hat{c}\hat{N}_c \geq \alpha_L \hat{F}_L + \alpha_D \hat{F}_D \tag{7}$$

which, taken at the equality, allows the footing to be designed,

$$B = \frac{\alpha_L \hat{F}_L + \alpha_D \hat{F}_D}{\Psi_u \varphi_{gu} \hat{c}\hat{N}_c} \tag{8}$$

Failure of the footing occurs if the actual total load on the footing, $F_T = F_L + F_D$, where F_L is the actual live load and F_D is the actual dead load (both random), exceeds the actual (random) resistance. The probability of failure is thus

$$p_f = \mathrm{P}\left[F_T > c_g N_{c_g} B\right] \tag{9}$$

where c_g and N_{c_g} are some sort of averages of the random cohesion and friction fields, taken in the vicinity of the footing, such that the product $c_g N_{c_g} B$ has the same distribution as the actual resistance of the spatially variable ground. Past research by the authors has shown that c_g and

N_{c_g} are well approximated by suitably selected geometric averages of c and ϕ in the vicinity of the foundation. Substituting eq. (8) into eq. (9) and collecting all random variables to the left side of the inequality leads to

$$p_f = \mathrm{P}\left[F_T \frac{\hat{c}\hat{N}_c}{c_g N_{c_g}} > \frac{\hat{F}_T}{\Psi_u \varphi_{gu}} \right] \tag{10}$$

If we let

$$W = F_T \frac{\hat{c}\hat{N}_c}{c_g N_{c_g}} \tag{11}$$

then the failure probability can be written in general terms (for either ULS or SLS by dropping the u subscript on the resistance and consequence factors) as

$$p_f = \mathrm{P}\left[W > \frac{\hat{F}_T}{\Psi \varphi_g} \right] \tag{12}$$

The random variables on the right-hand-side of eq. (11) are all assumed to be lognormally distributed. If this assumption is true, then W is also (at least approximately) lognormally distributed, so that

$$\begin{aligned} p_f &= \mathrm{P}\left[\ln W > \ln \hat{F}_T - \ln \Psi \varphi_g \right] \\ &= 1 - \Phi\left(\frac{\ln \hat{F}_T - \ln \Psi \varphi_g - \mu_{\ln W}}{\sigma_{\ln W}} \right) \end{aligned} \tag{13}$$

where Φ is the cumulative standard normal distribution function. Noting that the probability of failure can be expressed in terms of the reliability index, β, as $p_f = 1 - \Phi(\beta)$, then an explicit expression for the total factor applied to the resistance, is

$$\Psi \varphi_g = \frac{\hat{F}_T}{\exp\left\{ \mu_{\ln W} + \beta \sigma_{\ln W} \right\}} \tag{14}$$

Eq's (13) and (14) can be used to determine the failure probability and total resistance factor for many geotechnical problems, so long as suitable averaging regions can be found under or around the geotechnical system. It is expected that the only geotechnical problems which cannot be easily handled by eq's (13) and (14) are slope stability and problems where the soil acts as both the load and the resistance (e.g., some retaining wall systems). In addition, it is found that usually $\mu_{\ln W} = \mu_{\ln F_T}$ and that $\sigma_{\ln W}$ has a form which is common to most geotechnical problems.

For the bearing capacity of a strip footing, the parameters of the lognormally distributed random variable W are obtained by looking at the mean and variance of $\ln W$, where

$$\ln W = \ln F_T + \ln \hat{c} - \ln c_g + \ln \hat{N}_c - \ln N_{c_g} \tag{15}$$

Now assume that $\hat{c}$ and c_g are defined as geometric averages over the sample volume and over some suitable volume under/around the foundation, respectively. If so, then $\ln \hat{c}$ and $\ln c_g$ are arithmetic averages of $\ln c(\underset{\sim}{x})$,

$$\begin{aligned} \ln \hat{c} &= \frac{1}{V_s} \int_{V_s} \ln c(\underset{\sim}{x}) d\underset{\sim}{x} \\ \ln c_g &= \frac{1}{V_f} \int_{V_f} \ln c(\underset{\sim}{x}) d\underset{\sim}{x} \end{aligned} \tag{16}$$

where $\underset{\sim}{x}$ is spatial position, V_s is the volume of the soil sample, and V_f is the suitably selected volume of the averaging region in the vicinity of the foundation. The main difficulty with the solution of eq's (13) and (14) is with the selection of an appropriate averaging region, V_f .

In order to solve eq's (13) and (14), the mean and variance of $\ln W$ must be found. The mean is relatively simple if the ground is assumed to be statistically stationary (the mean and covariance structure remains constant over space), so that

$$\begin{aligned} \mu_{\ln \hat{c}} &= \mu_{\ln c_g} = \mu_{\ln c} \\ \mu_{\ln \hat{N}_c} &= \mu_{\ln N_{c_g}} = \mu_{\ln N_c} \end{aligned} \tag{17}$$

which gives

$$\mu_{\ln W} = \mu_{\ln F_T} \tag{18}$$

The variance of $\ln W$ is complicated by the random field model of the ground. As mentioned above, the basic idea is to replace the spatial variability of the actual ground with suitably defined local averages. Figure 2 illustrates the local averages involved: one local average under the footing is the region V_f , and if the size of V_f

is properly selected, then the ground properties averaged over V_f will have approximately the same bearing capacity distribution as the actual ground. Because the bearing failure follows the weakest path through the ground, a geometric average has been found to be appropriate (Fenton et al., 2008). Similarly, in order to perform the design, the ground will have been sampled at some location and then the characteristic ground parameters used in the design would be some sort of average of the sample values. If it is assumed that the soil sample is actually a CPT sounding of depth H at some location r away from the center of the footing, then the characteristic ground parameters would be an average of the observations over the volume V_s. It will be assumed here that a CPT sounding reflects the soil's strength parameters over a region around the cone of width Δx and it is further assumed that the appropriate average to use is again a geometric average.

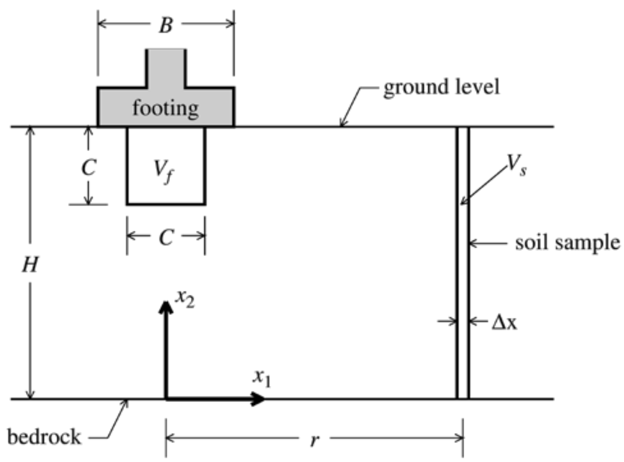

Figure 2. Averaging regions used to predict probability of bearing capacity failure.

If the load, F_T, and ground strength parameters, in this case c and ϕ, are assumed to be mutually independent then, to at least first order (see Fenton et al., 2008, for details),

$$\sigma^2_{\ln W} = \sigma^2_{\ln F_T} + \left(\sigma^2_{\ln c} + \sigma^2_{\ln N_c}\right)\left[\gamma_f + \gamma_s - 2\gamma_{fs}\right] \quad (19)$$

where γ_f is the variance reduction due to (geometric) averaging over a suitable region $\left(V_f\right)$ under or around the foundation, γ_s is the variance reduction due to (geometric) averaging over the soil sample volume (V_s), and γ_{fs} is the average correlation coefficient between the region V_f and the region V_s. The last is really a reflection of how well the soil sample describes the nature of the ground under the footing. As r increases, it is expected that γ_{fs} will decrease, indicating that the ground conditions at the footing are less well predicted by the sample. In this way, the degree of 'site understanding' that goes into the design of the footing can be reflected by adjusting r. If a designer has high confidence in their understanding of the ground parameters under the footing being designed, then that corresponds to a small value of r in this model. Conversely, low understanding of ground properties under the footing corresponds to a large value of r. In detail,

$$\gamma_f = \frac{1}{V_f^2}\int_{V_f}\int_{V_f} \rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, d\underset{\sim}{\eta}\, d\underset{\sim}{\xi}$$

$$\gamma_s = \frac{1}{V_s^2}\int_{V_s}\int_{V_s} \rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, d\underset{\sim}{\eta}\, d\underset{\sim}{\xi} \quad (20)$$

$$\gamma_{fs} = \frac{1}{V_f V_s}\int_{V_f}\int_{V_s} \rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, d\underset{\sim}{\eta}\, d\underset{\sim}{\xi}$$

where $\underset{\sim}{\eta}$ and $\underset{\sim}{\xi}$ are spatial positions and ρ returns the correlation coefficient between two points in the ground separated by distance $\underset{\sim}{\eta} - \underset{\sim}{\xi}$.

The averaging volume, V_s, is usually known, at least approximately, and will be one of the following values;

1) for 1-D averaging, $V_s = H$,
2) for 2-D averaging, $V_s = \Delta x \times H$,
3) for 3-D averaging, $V_s = \Delta x \times \Delta x \times H$.

The main challenge at this point is to decide on the appropriate size of the averaging volume, V_f. The geotechnical failure mechanism below (or around) the foundation usually involves some averaging of the strength or deformation properties of the ground and the size of V_f

should properly reflect the actual averaging. This means that V_f is dependent on the size of the foundation itself, which means that, strictly speaking, V_f is not known until after the foundation is designed (which means that the resistance factors need to be known before V_f can be determined).

In some cases, the variance reduction factor, γ_f, and the average correlation, γ_{fs}, are not very sensitive to fairly significant changes in V_f. This mean that V_f can sometimes be reasonably approximated by using a `typical' design, perhaps based on the mean ground properties and a typical (or traditional) resistance factor. In other cases, the variances and correlations are more sensitive to the size of V_f, in which case an iterative approach provides better results. If iteration is required, the basic algorithm to be used is as follows;

1) choose a reasonable starting value for the total resistance factor $(\Psi\varphi_g)$,
2) find the minimum foundation dimensions which satisfy the LRFD requirements (see eq. 3),
3) set the V_f averaging domain as some appropriate function of the foundation dimensions (this step will be discussed in more detail for each geotechnical problem considered shortly),
4) compute γ_f, γ_s, and γ_{fs} according to eq's (20),
5) use eq. (19) to compute $\sigma^2_{\ln W}$,
6) update the total resistance factor $(\Psi\varphi_g)$ according to eq. (14). Compute the failure probability, p_f, according to eq. (13) if desired. If the total resistance factor has changed by only within some relative error tolerance (e.g., 0.001), or if p_f is within some relative error tolerance from the target p_m, then the iterations can stop. Otherwise, repeat from step 2 using the adjusted value of the total resistance factor.

Once the total resistance factor, $(\Psi\varphi_g)$, has been determined for a variety of values of the target failure probability, p_m, the consequence factor, Ψ, is determined rather simply. Consider again the bearing capacity problem and assume that the total resistance factor has been determined for $p_m = 1/1000$ (low consequence), $p_m =$ 1/500 (typical consequence) and $p_m = 1/10000$ (high consequence). Denoting the corresponding total resistance factors $(\Psi_u\varphi_{gu})_{low}$, $(\Psi_u\varphi_{gu})_{typ}$, and $(\Psi_u\varphi_{gu})_{high}$, then assuming that $\Psi_u = 1.0$ for the typical case, we get

$$\begin{aligned}\text{low consequence: } \Psi_u &= \left(\Psi_u\varphi_{gu}\right)_{low} / \left(\Psi_u\varphi_{gu}\right)_{typ} \\ \text{high consequence: } \Psi_u &= \left(\Psi_u\varphi_{gu}\right)_{high} / \left(\Psi_u\varphi_{gu}\right)_{typ}\end{aligned} \tag{21}$$

3. FACTORS FOR THE ULS DESIGN OF SHALLOW FOUNDATIONS

To illustrate the above theory, the required resistance and consequence factors for the ULS bearing capacity design of a shallow foundation, with parameters as given in Table 1, will be considered. The characteristic factored load, $\hat{F}_T$, assumes live and dead load factors of $\alpha_L = 1.5$ and $\alpha_D = 1.25$ along with live and dead load bias factors of 1.41 and 1.18, respectively.

Table 1: Parameters used in the investigation of required resistance and consequence factors for the ULS design of shallow foundations.

Parameter	Value
μ_c, v_c	100 kN/m, 0.3
$\phi_{min}, \phi_{max}, s$	$10^o, 30^o, 3$
μ_L, v_L	200 kN/m, 0.3
μ_D, v_D	600 kN/m, 0.15
$\hat{F}_T$	1308 kN/m
$\Delta x, H$	0.15 m, 4.8 m
θ	0.1 to 50 m

The main features of the ULS reliability-based design of a shallow foundation can be found in Fenton et al. (2008). They found that V_f is well approximated by a square of dimension $C \times C$ centered under the footing (see Figure 2), where C is about 80% of the mean depth of the classical wedge failure zone given by Prandtl,

$$C = \frac{0.8}{2} \hat{\mu}_B \tan\left(\frac{\pi}{4} + \frac{\mu_\phi}{2}\right) \tag{22}$$

In the above, $\hat{\mu}_B$ is an estimate of the mean footing width obtained by evaluating eq. (8) at the mean of the ground properties

$$\hat{\mu}_B = \frac{\hat{F}_T}{0.7\mu_c \mu_{N_c}} \tag{23}$$

Using this result in eq. (23) to define $V_f = C \times C$ allows the results of the previous section to be used to find the failure probability and resistance factors required to achieve a target failure probability, p_m.

For the uncertainty levels given in Table 1, the resistance factors required to achieve a typical lifetime maximum acceptable failure probability of $p_m = 1/5000$ are shown in Figure 3. Notice the presence of a ‘worst case’ correlation length which is approximately equal to the distance between the foundation and the sample, *r*.

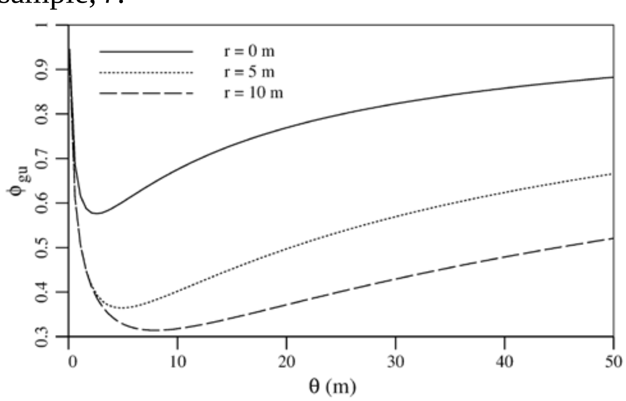

Figure 3. Resistance factors required to achieve $p_m = 1/5000$ ($\beta = 3.5$) for the bearing capacity design of a shallow foundation ($v_c = 0.3$, and conservative load bias factors).

Figure 4 shows the low and high consequence factors, obtained using eq's (21). Although the consequence factor is supposed to be primarily dependent on the target maximum acceptable failure probability appropriate for the failure consequence, there is some residual dependence on site understanding (*r*) and correlation length (θ). However, the dependence is slight, amounting to less than 4% relative change for high consequence (Figure 4b) and less than 12% for low consequence (Figure 4a). This dependence on *r* and θ is negligible compared to the changes seen in the resistance factor, which is supposed to depend on *r* and θ, (see φ_{gu} in Figure 3) of up to 300%.

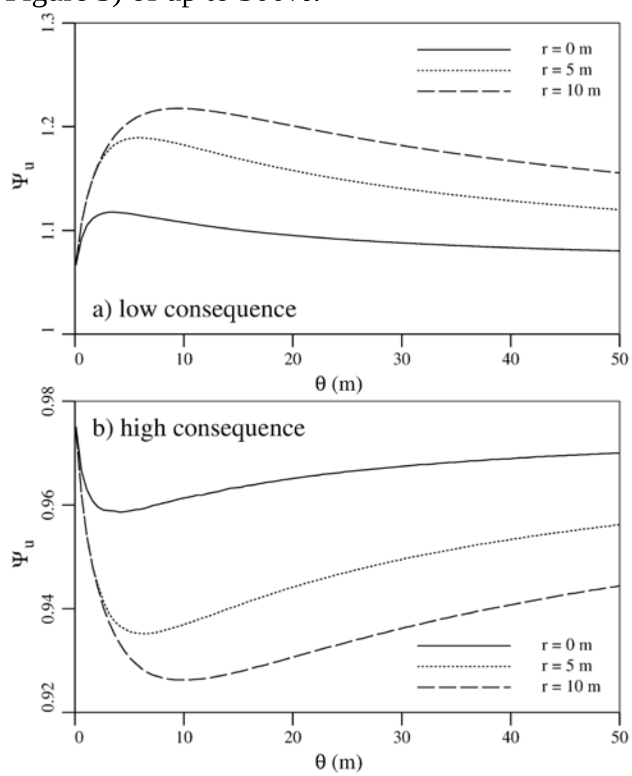

Figure 4. ULS consequence factors for shallow foundations required to adjust $p_m = 1/5000$ to low consequence $p_m = 1/1000$ in (a) and to high consequence $p_m = 1/1000$ in (b).

Noting that consequence factors of lower value result in lower failure probability, it can be seen that if Ψ_u is selected as 0.9 for high failure consequence cases, then the target maximum acceptable failure probability will be less than $p_m = 1/10,000$ for all cases of r and θ considered in Figure 4b.

Since it is not so important to remain conservative when the failure consequences are already low, Figure 4a suggests that $\Psi_u = 1.15$ might be appropriate for low failure consequence designs.

4. CONCLUSIONS

The paper presents a unified theory which allows the estimation of both failure probability and the resistance and consequence factors required to achieve a target failure probability. Perhaps the most important component of this unified theory is eq. (19), which in a more generalized form appears as

$$\sigma^2_{\ln W} = \sigma^2_{\ln F_T} + \sigma^2_{\ln R}\left[\gamma_f + \gamma_s - 2\gamma_{fs}\right] \quad (24)$$

where R denotes 'resistance' and is replaced by the ground parameter(s) which are important for the problem. This equation can then be used in eq. (13), to determine failure probability, or in eq. (14) to determine required resistance factors given a target reliability. Eq. (24) includes the following components;

1) variability of the applied load $(\sigma_{\ln F_T})$,
2) variability of the ground $(\sigma_{\ln R})$,
3) variance reduction due to averaging of the ground properties under and around the foundation (γ_f),
4) variance reduction due to averaging of the ground properties found in the soil sample (γ_s), and perhaps most importantly,
5) correlation between the sample and the properties of the ground under and around the foundation (γ_{fs}).

The last allows for a reasonable modeling of 'site understanding' so that resistance factors can be selected based on how well the response of the ground supporting the foundation can be predicted. The distance r used in this study can be used as a proxy to reflect general site and model understanding, where `model understanding' refers to how accurate the ground response prediction model is. As site and model understanding decreases, the corresponding value of r selected in this study would be increased.

The consequence factor is used to adjust the target failure probability from the 'typical' level to either a high or low consequence level. Although not shown in this paper, a review of the consequence factors required for various limit states shows that the consequence factors are very similar, meaning that they are largely independent of the limit state under consideration. Thus, the distinction between Ψ_s and Ψ_u can be dropped, and a common consequence factor, Ψ, used.

5. ACKNOWLEDGEMENTS

The authors are thankful for the support provided by the Natural Sciences and Engineering Research Council of Canada and by a Special Research Project Grant from the Ministry of Transportation of Ontario (Canada).

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