# Role of Soil and Structural Heterogeneity in Geotechnical System Redundancy

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**Abstract.** It is well known that redundancy generally improves system reliability. For example, a geotechnical support system comprised of a single monopile will have the same failure probability as the monopile itself. Alternatively, if the geotechnical support system is comprised of two piles, each of which can support the load with probability $1-p_f$, then the system failure probability will lie somewhere between $p_f$ and $p_f^2$. In this case, if the failure probability of an individual pile is $p_f = 1/100$, then the system failure probability will lie between 1/100 and 1/10,000, depending on the degree of statistical dependence between the piles. Clearly, redundancy in the geotechnical system has the potential to significantly reduce the system failure probability. From a design point of view, since redundancy generally increases system reliability, the individual system elements (e.g., piles) need not necessarily be designed to the same level of reliability. In other words, if the supported load is distributed amongst a number of footings or piles, redundancy should be taken into account to achieve construction savings while maintaining overall safety. This paper looks specifically at the effects of redundancy in pile support systems on the overall system reliability. It is assumed that the support only fails when all piles have failed and piles fail randomly according to the local ground strength (pile structural capacity is not considered). Two load transfer models between failed (excessively displaced) and surviving piles are considered. Pile system reliability is then estimated as a function of the distribution of pile resistance, the load transfer model, the number of piles, and the target design reliability of individual piles. Charts are produced to allow the selection of individual target design reliability for a given number of piles and the target system reliability.

**Keywords.** System redundancy, pile systems, deep foundations, pile reliability, system reliability

## 1. Introduction

System redundancy generally increases system reliability, which means that individual elements (e.g. piles) need not be designed to the same level of reliability if it is known that failure of individual piles will not result in system failure. In other words, if the load is distributed amongst a number of piles, redundancy should be accounted for to achieve construction savings while maintaining overall safety.

All systems are made up of one or more components, and the reliability of a system largely depends on the reliability of the individual components. Thus, it is necessary to first identify the components that make up a system, and how they individually contribute to system reliability. The next step is to establish the distribution of the individual component reliabilities from which reliability of the system is computed.

The main goal of this paper is to investigate the relationship between number of piles and system reliability for various resistance statistics and various levels of dependency between piles.

## 1. Methodology

Piles are often used in groups and connected at the top by a pile cap. A pile system consisting of $n_p$ piles is considered here, which supports the total vertical load $F_T$. Practically speaking, the following three scenarios can be considered when reliability of a pile system is of concern:

- Approach 1: The supported structure is somewhat flexible, so that when a pile is displaced past its ultimate capacity, its load is distributed amongst the remaining piles (load sharing). System failure occurs when all piles are displaced beyond their ultimate capacities.

- Approach 2: The supported structure is rigid, in which case the piles are all displaced equally by the applied total load. This implies that the foundation reaction is purely the sum of the individual pile resistances, where the resistance of each pile depends on its common imposed displacement.

- Approach 3: Piles act independently and are loaded independently. It is assumed in this case that the supported structural performance is lost (system failure) if any of the piles fail. System failure thus occurs if one or more piles fail, i.e,

$$p_f = \mathrm{P}\left[ F_T / n_p \geq \min_i R_i \right] \quad (1)$$

where $R_i$ is the resistance provided by the $i^{th}$ pile.

Since approach 3 does not involve any redundancy, it will not be considered further in this research. Therefore, only the first two approaches are investigated in detail here.

*2.1. Approach 1*

Assuming that $F$ is the event corresponding to failure of the pile system, then

$$F = \left\{ F_1 \cap F_2 \cap ... \cap F_{n_p} \right\} \quad (2)$$

That is, the failure of the pile system occurs only if all of its elements fail, where $F_i$ denotes the failure of the $i^{th}$ pile, $i = 1, 2, ..., n_p$. Thus, the failure probability of the pile system can be computed using the multiplication rule as

$$\begin{aligned} p_f &= \mathrm{P}[F] = \mathrm{P}\left[ F_1 \cap F_2 \cap ... \cap F_{n_p} \right] \\ &= \mathrm{P}[F_1]\mathrm{P}[F_2 \mid F_1]\mathrm{P}[F_3 \mid F_1 \cap F_2]... \\ &\quad \mathrm{P}\left[ F_{n_p} \mid F_1 \cap F_2 ... \cap F_{n_p - 1} \right] \end{aligned} \quad (3)$$

Assuming that $R_i$ is the resistance provided by $i^{th}$ pile, then

$$p_f = \mathrm{P}[F_i] = \mathrm{P}\left[ R_i < F_T / n_p \right] \quad (4)$$

where all piles are assumed to have the same resistance distribution (with the same mean and variance).

The first probability in Eq. (3) is calculated as follows. Assuming that the lognormal distribution is the appropriate distribution to represent both load and resistance, then the probability that the first pile fails under load $F_T / n_p$ is

$$\begin{aligned} \mathrm{P}[F_1] &= \mathrm{P}\left[ R_1 < F_T / n_p \right] = \mathrm{P}\left[ n_p R_1 / F_T < 1 \right] \\ &= \mathrm{P}\left[ \ln\left( n_p R_1 / F_T \right) < 0 \right] = \mathrm{P}\left[ \ln Z_1 < 0 \right] \\ &= \Phi\left( \frac{-\mu_{\ln Z_1}}{\sigma_{\ln Z_1}} \right) \end{aligned} \quad (5)$$

where

$$Z_1 = n_p R_1 / F_T \quad (6)$$

is lognormally distributed so that

$$\ln Z_1 = \ln\left( n_p \right) + \ln R_1 - \ln F_T \quad (7)$$

is normally distributed with mean and variance

$$\begin{aligned} \mu_{\ln Z_1} &= \ln(n_p) + \mu_{\ln R} - \mu_{\ln F_T} \\ \sigma^2_{\ln Z_1} &= \sigma^2_{\ln R} + \sigma^2_{\ln F_T} = \ln\left( 1 + v_R^2 \right) + \ln\left( 1 + v_T^2 \right) \end{aligned} \quad (8)$$

assuming independence between load and resistance. $v_R = \sigma_R / \mu_R$ and $v_T = \sigma_T / \mu_T$ are the coefficients of variation of $R_1$ and $F_T$ respectively.

One simple way to introduce dependence between piles is to assume that they share the load $F_T$ equally, but otherwise fail independently. This means that if one pile fails, the other $n_p - 1$ piles share the load $F_T / (n_p - 1)$.

The second probability in Eq. (3) corresponds to the case that a pile fails given that another pile has already failed. In this case, the total load $F_T$ is assumed supported by, and distributed amongst the remaining $n_p - 1$ piles, so that

$$\begin{aligned} \mathrm{P}\left[F_2 \mid F_1\right] &= \mathrm{P}\left[R_2 < F_T / (n_p - 1)\right] \\ &= \mathrm{P}\left[(n_p - 1) R_2 / F_T < 1\right] \\ &= \mathrm{P}\left[\ln((n_p - 1) R_2 / F_T) < 0\right] \\ &= \mathrm{P}\left[\ln Z_2 < 0\right] = \Phi\left(\frac{-\mu_{\ln Z_2}}{\sigma_{\ln Z_2}}\right) \end{aligned} \quad (9)$$

where

$$Z_2 = (n_p - 1) R_2 / F_T \quad (10)$$

is lognormally distributed so that

$$\ln Z_2 = \ln\left(n_p - 1\right) + \ln R_2 - \ln F_T \quad (11)$$

is normlly distributed with mean

$$\mu_{\ln Z_2} = \ln(n_p - 1) + \mu_{\ln R} - \mu_{\ln F_T} \quad (12)$$

and variance as given by Eq. (8). In general, the probability that a pile fails given that $i-1$ piles have already failed is

$$\begin{aligned} &\mathrm{P}\left[F_i \mid F_1 \cap F_2 ... \cap F_{i-1}\right] \\ &= \mathrm{P}\left[\ln\left((n_p - i + 1) R_i / F_T\right) < 0\right] \\ &= \mathrm{P}\left[\ln Z_i < 0\right] = \Phi\left(\frac{-\mu_{\ln Z_i}}{\sigma_{\ln Z_i}}\right) \end{aligned} \quad (13)$$

where

$$Z_i = (n_p - i + 1) R_i / F_T \quad (14)$$

is lognormal so that

$$\ln Z_i = \ln\left(n_p - i + 1\right) + \ln R_i - \ln F_T \quad (15)$$

is normal with mean

$$\mu_{\ln Z_i} = \ln\left(n_p - i + 1\right) + \mu_{\ln R} - \mu_{\ln F_T} \quad (16)$$

and variance as given by Eq.(8).

Substituting Eq. (13) into Eq. (3) results in

$$p_f = \mathrm{P}\left[F\right] = \prod_{i=1}^{n_p} \Phi\left(\frac{-\mu_{\ln Z_i}}{\sigma_{\ln Z_i}}\right) \quad (17)$$

Once the probability of the system failure is determined, the number of piles, $n_p$, to achieve a certain target system failure probability, $p_{\max}$, can be found using root-finding algorithms, such as one-point iteration or bisection, by setting $p_f = p_{\max}$. Now $n_p$ can be found as roof of

$$p_{\max} - \prod_{i=1}^{n_p} \Phi\left(\frac{-\mu_{\ln Z_i}}{\sigma_{\ln Z_i}}\right) = 0 \quad (18)$$

Similarly, for a given $p_{\max}$, $n_p$, $v_R$, $v_T$, and $\mu_T$, the required individual mean pile resistance, can be found by using the bisection algorithm to solve for the required $\mu_{\ln R}$ in Eq. (18) combined with Eq. (16). From this, the required pile mean resistance, $\mu_R$, can be found using the following transformation,

$$\mu_R = e^{\mu_{\ln R}} \sqrt{1 + v_R^2} \quad (19)$$

The system reliability index, $\beta_{sys}$ is defined as

$$\beta_{sys} = \Phi^{-1}\left(1 - p_{\max}\right) = -\Phi^{-1}\left(p_{\max}\right) \quad (20)$$

where $p_{\max}$ is the target failure probability. For a given $\beta_{sys}$, the reliability index for an individual pile, $\beta_i$ can be obtained by first finding required $n_p$ or $\mu_{\ln R}$, using the methods described above, then calculating individual pile failure using Eq. (4), and finally

$$\beta_i = -\Phi^{-1}\left(\mathrm{P}\left[F_i\right]\right) \quad (21)$$

### *2.2. Approach 2*

If the supported structure is rigid, then any displacement of the structure involves an equal displacement of each pile. This means that the resistance provided by each pile is obtained from its load-displacement curve at a given common displacement. Each pile will have a different load-displacement curve due to variations in soil property, installation procedures, etc, so that the resistance provided by each pile will be a random variable. Let $R_i$ be the resistance provided by the $i^{th}$ pile at the displacement imposed by the structure, so that the total resistance will be

$$R = \sum_{i=1}^{n_p} R_i \quad (22)$$

The problem now is to determine the distribution of $R_i$. General approaches are to statistically analyze individual pile tests and develop distribution fits. When pile groups are involved, correlation between piles can affect the overall pile system resistance. This phenomenon is typically handled by using the individual pile test results to develop the distributions and introducing a system efficiency factor, which adjusts the overall pile system resistance to account for this correlation. The total resistance thus becomes

$$R = \xi \sum_{i=1}^{n_p} R_i \qquad (23)$$

where $\xi$ is a system efficiency factor (typically $\leq 1$), and is defined as the ratio of the ultimate resistance of a pile system to the sum of the resistances of the individual piles. The failure probability of the pile system is calculated as

$$p_f = \mathrm{P}\left[ F_T > \xi \sum_{i=1}^{n_p} R_i \right] = \mathrm{P}\left[ F_T - \xi \sum_{i=1}^{n_p} R_i > 0 \right] \qquad (24)$$

Since sums of random variables tend to the normal distribution by the central limit theorem, the sum of individual resistance will be at least approximately normal assuming some independence between pile resistances. For simplicity, it will be assumed that the total load (also being a sum of individual structural loads) is also at least approximately normal. This means that the quantity $Y = F_T - \xi \sum_{i=1}^{n_p} R_i$ is normally distributed with mean and variance

$$\begin{aligned} \mu_Y &= \mu_T - \xi n_p \mu_R \\ \sigma_Y^2 &= \sigma_T^2 + \xi^2 n_p \sigma_R^2 \end{aligned} \qquad (25)$$

So that

$$p_f = \mathrm{P}\left[ Y > 0 \right] = 1 - \Phi\left( \frac{-\mu_Y}{\sigma_Y} \right) = \Phi\left( \frac{\mu_Y}{\sigma_Y} \right) \qquad (26)$$

Eq. (25) assumes that the individual pile resistances are uncorrelated. In general, this will not be true, and leads to a variance which is lower than if the pile resistances are positively correlated. The implication of correlation between piles has been discussed previously in this section and has been assumed to be handled by the system efficiency factor. Thus, it will be assumed here that the pile resistances are uncorrelated.

For a given target maximum failure probability, $p_{\max}$, the required number of piles, $n_p$, can be found by solving

$$p_{\max} - \Phi\left( \frac{\mu_T - \xi n_p \mu_R}{\sqrt{\sigma_T^2 + \xi^2 n_p \sigma_R^2}} \right) = 0 \qquad (27)$$

for $n_p$. Alternatively, Eq. (27) can be solved for $\mu_R$, for a given $p_{\max}$ and $n_p$, to find the required individual pile resistance to achieve a given $p_{\max}$. Once $n_p$ or $\mu_R$ are obtained for a certain $p_{\max}$ associated with system reliability $\beta_{sys} = -\Phi^{-1}\left( p_{\max} \right)$, then the individual pile failure probability and in turn the individual pile reliability, $\beta_i$, can be found from

$$\begin{aligned} p_{f_i} &= \mathrm{P}\left[ F_T / n_p > R_i \right] = \mathrm{P}\left[ F_T / n_p - R_i > 0 \right] \\ &= \Phi\left( \frac{\mu_T / n_p - \mu_R}{\sqrt{\sigma_T^2 / n_p^2 + \sigma_R^2}} \right) = \Phi\left( -\beta_i \right) \end{aligned} \qquad (28)$$

## 3. Results

The effect of number of piles on system failure probability is depicted in Figures 1 and 2 for $\mu_T = 50$ kN, $v_T = 0.1$, and $v_R = 0.3$, and various mean resistances, for the two approaches. When compared to Figure 1, Figure 2 demonstrates the need for a higher number of piles to support the applied (random) load in approach 2.

Figures 3 and 4 illustrate how the reliability of individual piles, $\beta_i$, relates to the system reliability, $\beta_{sys}$. For a given $\mu_T$, $v_T$, $v_R$, and $n_p$, $\beta_i$ is determined in Figure 3 by solving Eq. (18) for $\mu_{\ln R}$, finding $\mu_R$ via Eq. (19), and finally calculating $\beta_i$ using Eq. (21). Similarly, $\beta_i$ values in Figure 4 are obtained by first solving Eq. (27) for $\mu_R$, for a given $\mu_T$, $v_T$, $v_R$, and $n_p$, and then calculating $\beta_i$ via Eq. (28). These figures can be used for design by drawing a vertical line at the target system reliability index, $\beta_{sys}$, and then reading off the required $\beta_i$ for a given $n_p$. For example, for a moderate system reliability $\beta_{sys}$ = 3.5 corresponding to $p_f$ = 1/5,000 using approach 1, the required single pile reliability index ranges from $\beta_i = 0.9$ for $n_p = 20$ to 2.15 for $n_p = 5$, which corresponds to individual pile failure probabilities ranging from $p_f$ = 0.18 to 0.016. Approach 2 recommends a similar but a narrower range for $\beta_i$ (between 1.1 and 1.7) as shown in Figure 4.

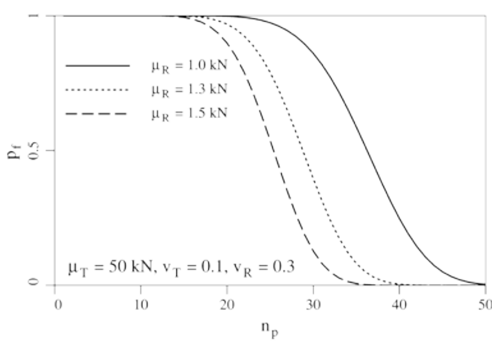


**Figure 1.** Plot of $p_f$ versus $n_p$ for $\mu_T = 50$ kN, $v_T = 0.1$, and $v_R = 0.3$, and various mean resistance values $\mu_R$ = 1.0, 1.3, and 1.5 kN, generated by Eq. (17) using approach 1

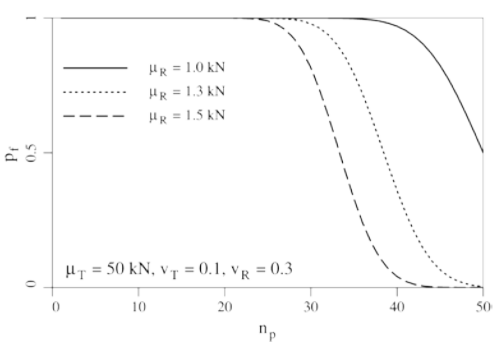


**Figure 2.** Plot of $p_f$ versus $n_p$ for $\mu_T = 50$ kN, $v_T = 0.1$, and $v_R = 0.3$, and various mean resistance values $\mu_R$ = 1.0, 1.3, and 1.5 kN, generated by Eq. (26) using approach 2 ( $\xi = 1$ )

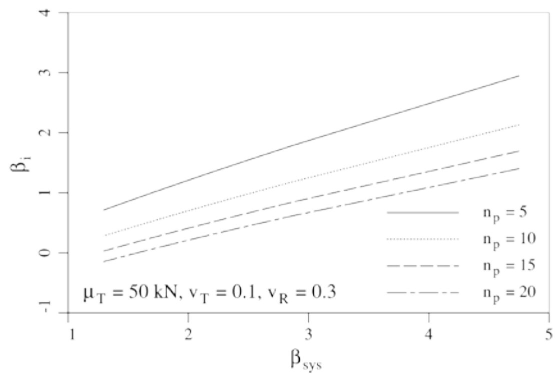


**Figure 3.** Plot of $\beta_i$ versus $\beta_{sys}$ for $\mu_T = 50$ kN, $v_T = 0.1$, and $v_R = 0.3$, and various number of piles, $n_p$, generated by Eq's. (18), (19), and (21) using approach 1

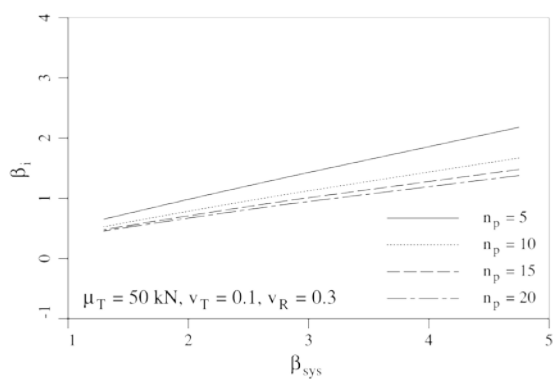


**Figure 4.** Plot of $\beta_i$ versus $\beta_{sys}$ for $\mu_T = 50$ kN, $v_T = 0.1$, and $v_R = 0.3$, and various number of piles, $n_p$, generated by Eq's. (27) and (28) using approach 2 ( $\xi = 1$ )

Figure 5 compares the two approaches in terms of the required individual reliability index, $\beta_i$, recommended for a given target reliability index $\beta_{sys}$. It is observed that the values of $\beta_i$ generated by approach 2 with $\xi = 1$ fall inside the range generated by approach 1.

In general, a reliability index of $\beta_i$ = 3.0 ( $p_f$ = 1/1,000) is prescribed in geotechnical design practice to target the design of an individual pile in non-redundant pile systems ( $n_p \leq 4$ ), and $\beta_i$ = 2.3 ( $p_f$ = 1/100) for redundant pile systems ( $n_p \geq 5$ ) (Zhang et al. 2001, Paikowsky et al., NCHRP, 2004, Allen 2005, and Barker et al., NCHRP, 1991). According to Zhang et al. (2001), a $\beta_{sys}$ value of 3.0 requires a $\beta_i$ = 2.0 to 2.8. Figure 5 gives a $\beta_i$ = 0.7 to 1.9 for $\beta_{sys}$ = 3.0, when $n_p$ ranges from 5 to 20 over both approaches.

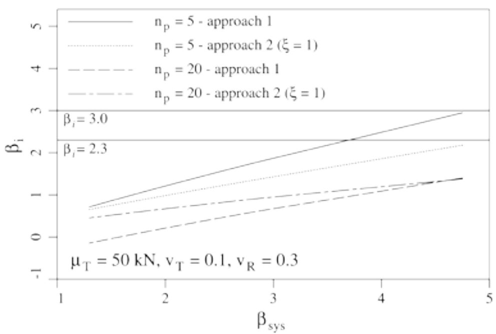


**Figure 5**. Comparison of two approaches in terms of $\beta_i$ versus $\beta_{sys}$ for $n_p = 5$ and 20 and $\xi = 1$

Evidently, Zhang et al.'s (2001) results are more conservative than suggested here. It is not possible to carefully investigate the cause of the discrepancy because Zhang et al. (2001) do not clearly describe their model. It is felt that perhaps some of the difference is due to natural tendencies towards conservatism in designing individual piles and to the fact that piles do not actually fail independently. More research is required to investigate the effects of correlation between pile resistances.

## 4. Conclusions

In this paper, the reliability of a pile system for various levels of pile redundancy and resistance statistics, are studied, and a relationship between reliability of a pile system and its individual components is established. For a given pile system reliability, the reliability of an individual pile is determined analytically for two different loading scenarios. The individual piles can then be designed to achieve the individual pile reliability required to maintain the target system reliability. The redundancy reliability model is simple and easy to implement. At the moment, it assumes that the individual pile resistances are independent, which is probably not generally true. An improved model would include the correlation between pile resistances (through the soil) and its impact on reliability of a pile system.


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