Target Geotechnical Reliability for Redundant Foundation Systems

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ABSTRACT
Geotechnical support systems (e.g., deep and shallow foundations) generally involve at least some redundancy. For example, if a building is supported by *n* separate foundations, then failure (e.g., excessive settlement) of a single foundation will generally not result in failure of the building if the building is able to shed the load from the failed foundation to adjacent foundations. This load shedding ability lends the foundation system redundancy -- system failure only occurs if multiple foundations fail. This paper investigates the relationship between geotechnical redundancy and system reliability using Monte Carlo simulation and presents a means to relate individual foundation reliability to system reliability for the purpose of establishing design criteria.

RÉSUMÉ
Systèmes de soutien géotechniques (par exemple, des fondations profondes et peu profondes) impliquent généralement au moins une certaine redondance. Par exemple, si un bâtiment est pris en charge par *n* fondations séparées, alors l'échec (par exemple, un tassement excessif) d'une seule fondation ne sera généralement pas entraîner une défaillance du bâtiment si le bâtiment est en mesure d'apporter la charge de la fondation a échoué aux fondations adjacentes . Ce délestage capacité confère à la redondance du système de fondation - défaillance du système ne se produit que si plusieurs fondations échouent. Cet article étudie la relation entre la redondance géotechnique et la fiabilité du système en utilisant la simulation et présente un moyen de relier la fondation individuelle fiabilité à la fiabilité du système dans le but d'établir des critères de conception.

1 INTRODUCTION

Piles are usually used in groups, and are designed with some level of redundancy to ensure that partial failure of the group doesn't result in the collapse of the entire system. Providing redundancy is costly and an economical approach to designing individual piles is of high interest in geotechnical and structural engineering. This paper investigates the reliability of a pile system, made up of n_p piles, for various number of piles and load and resistance statistics, and establishes a relationship between the reliabilities of a pile system and its individual components. Thus, for a target pile system reliability, and considering correlations between individual pile loads and resistances, the reliability of an individual pile can be determined using the simulation results presented here. Individual piles can then be designed so that they collectively achieve the required pile system reliability. In other words, a key question to answer is: at what level of reliability, β_i, should individual piles be designed to successfully achieve a target system reliability β_{sys}?

Reliability of multi-component redundant systems has been studied by numerous researchers for several decades now. A common technique is load-sharing in which, as components fail one by one, the total load applied to the system is redistributed amongst the surviving components. The load-sharing approach may be classified into equal load-sharing, tapered load-sharing, local load-sharing, nearest-neighbor load-sharing, and hybrid load-sharing (Durham et al., 1997). For instance, the Daniels system (Daniels 1945) assumes equal load-sharing, where all components are assumed to share equal parts of the total load. The Daniels system assumes a set of n_p components having independent and identically distributed resistances, R_i , $i = 1,...,n_p$, subjected to random total load F_T .

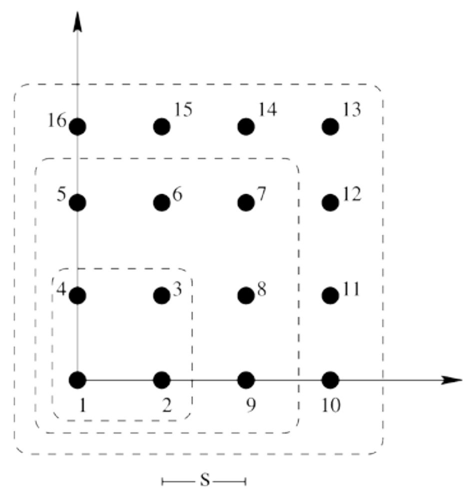

Figure 1: plan view of pile locations

Generally speaking, independent and identically distributed resistances will not realistically represent a multi-component pile system. In addition, the loads applied to each pile are often not equal, depending, as they do, on the pile settlement and the stiffness of the supported structure. Recognizing this, the redundancy model employed here introduces correlation amongst both the random resistances, R_i, and the loads, F_i, $i = 1, \ldots, n_p$, where n_p is the number of piles in the foundation system. Figure 1 illustrates the arrangement of the piles for cases where $n_p = 4, 9,$ and 16, with s denoting the pile spacing. The piles are characterized by individual resistances, R_i, supporting individual loads F_i. Note that while the case where $n_p = 4$ isn't usually considered to comprise a redundant pile system, it is included in this study as a lower bound on redundancy.

If a pile fails, some or all of its load, F_i, is assumed to be shared equally amongst the remaining surviving piles. Note that even after a pile has reached its ultimate capacity, R_i, it may continue to provide some residual resistance which may be less than R_i in the worst case. The residual resistance depends on the entire load-displacement curve associated with a pile, particularly for displacements beyond that corresponding to the ultimate capacity. For example, suppose that a realization of the random capacity of a pile is $R_i = 20$ kN, but that the realization of the random load applied to the pile is $F_i = 25$ kN. In this case the pile is overloaded by 5 kN and will displace into the ground as a result. As the pile settles, an increasing proportion of the original load will be redistributed to adjacent piles, the actual proportion being dependent on the stiffness of the supported structure and the nature of the load-displacement curve associated with the pile. It is beyond the scope of this preliminary study to also model the supported structural stiffness as well as the load-displacement behaviour of the pile. To simplify things, the residual resistance provided by the pile, once it has failed (i.e., $R_i < F_i$), is assumed to be $\left(1-a\right)R_i$. The parameter a gives the fraction of the resistance that is *lost* once the pile's ultimate capacity has been exceeded; a is 0.0 if the load-resistance curve for the pile becomes completely flat after the ultimate capacity, R_i, has been reached (i.e., the ultimate pile capacity is sustained regardless of additional pile displacement into the ground) and is 1.0 if all of the pile capacity is lost upon failure (the pile `plunges', as may occur in some clays). In some cases, e.g., in sand, the resistance continues to increase with displacement, in which case $a < 0$. This latter case is not considered here because it is unconservative – the reserves in strength are generally unknown and may correspond to displacements well beyond failure of the supported structure. It is assumed in this paper that the load displacement curve becomes at best flat ($a = 0$) and at worst plunges to zero after achieving its ultimate capacity. The actual value of a depends on the pile's load-displacement curve and the stiffness of the supported structure.

In general, if m out of n_p piles fail, then the remaining $n_p - m$ piles each support their initial applied load, F_i, as well as the excess load due to failure of the m piles, ΔF, that is

$$\Delta F = \frac{1}{n_p - m}\sum_{j=1}^{m}\left[F_j - (1-a)R_j\right] \qquad [1]$$

where it is assumed in the above that the piles have been numbered (or sorted) so that the first $j = 1, \ldots, m$ piles have failed (i.e., $R_j < F_j$ for $j = 1, \ldots, m$). The revised load on the i^{th}, $i = m+1, \ldots, n_p$, unfailed pile then becomes

$$F_i' = F_i + \Delta F \qquad [2]$$

Note that it is assumed here that the load which is shed from failed piles is shared equally between all remaining piles in the foundation system. That is, the stiffness of the supported structure is assumed to be such that it leads to equal load sharing of the loads not supported by all failed piles. Note also that the fraction of ultimate resistance lost by each pile after its failure (i.e., $R_i < F_i$), given by the parameter a, is assumed to be the same for all piles.

With the above in mind, this paper examines the reliability of a pile system for various levels of pile redundancy and load and resistance statistics using simulation. A relationship between the reliability of a pile system and the reliability of its individual components is established. The results can be used in cost-effectively, yet safely, designing individual piles to achieve a target system reliability index, β_{sys}.

The paper is organized as follows: In Section 2, a random field model is presented for a system of n_p spatially distributed piles. The corresponding simulation model is described in Section 3. The reliability-based pile design approach is discussed in Section 4 and the results are presented in Section 5. Conclusions are summarized in Section 6.

2 RANDOM FIELD MODEL

A random field $X(\underset{\sim}{t})$ is a collection of random variables $X_1 = X(\underset{\sim}{x}_1), X_2 = X(\underset{\sim}{x}_2), \ldots$, whose values are associated with each spatial location $\underset{\sim}{x}$. Values in a random field are usually spatially correlated, and the spatial dependence in a field is characterized by the field correlation structure, which is commonly specified through a correlation function parameterized by correlation length, θ. In this paper, an isotropic exponentially decaying Markov correlation function is used, defined by

$$\rho(\tau_{ij}) = \exp\left\{\frac{-2\left|\tau_{ij}\right|}{\theta}\right\} \qquad [3]$$

where τ_{ij} is the distance between any two points, X_i and X_j, in the field, and θ is the correlation length (Fenton and Griffiths, 2008).

A lognormal distribution is commonly used for modeling engineering properties due to its non-negative nature and its simple relationship with the normal distribution. In particular, a lognormal random field can be easily produced through a simple transformation of a

Gaussian random field. In general, if X is lognormal with mean and standard deviation μ_X and σ_X, then $\ln X$ is normal with parameters

$$\begin{aligned}\sigma^2_{\ln X} &= \ln(1+v_X^2) \\ \mu_{\ln X} &= \ln(\mu_X) - \frac{1}{2}\sigma^2_{\ln X}\end{aligned} \qquad [4]$$

where $v_X = \sigma_X / \mu_X$ is the coefficient of variation of X. In this research, both load, F, and resistance, R, are assumed to be lognormally distributed random variables. This implies that $\ln F$ and $\ln R$ are both normally distributed with parameters given by Eq. 4 (where the subscript X is suitably replaced by either R or F). Furthermore, both load and resistance are spatially varying random variables with an additional parameter being the correlation length, $\theta_{\ln F}$ and $\theta_{\ln R}$ respectively, replacing θ in Eq. 3.

3 SIMULATION MODEL

Various random field generation algorithms exist of which the Covariance Matrix Decomposition (CMD, see e.g., Fenton and Griffiths, 2008) method is employed in this research to provide realizations of the random load and resistance fields. CMD is an exact method of producing realizations of a discrete random field (i.e., at the pile locations) using the mean, $\mu_{\ln X}$, and covariance matrix, $\underset{\approx}{C}$, having elements $C_{ij} = \rho_{ij}\sigma_{\ln Xi}\sigma_{\ln Xj}$, $i, j = 1,2,...,n_p$ which give the covariance between any pair of points in the field separated by lag distance τ_{ij}, where $\rho_{ij} = \rho\left(\tau_{ij}\right)$ (see Eq. 3). For a stationary random field, having spatially constant variance, the covariance matrix $\underset{\approx}{C}$ is composed of the elements

$$C_{ij} = \begin{cases} \sigma^2_{\ln X} & i = j \\ \sigma^2_{\ln X}\rho_{ij} & i \neq j \end{cases} \qquad [5]$$

Since $\underset{\approx}{C}$ is a positive definite covariance matrix having elements C_{ij}, then a normally distributed (Gaussian) random field $G_i = G(\underset{\sim}{x}_i)$ can be produced according to

$$\underset{\sim}{G} = \underset{\sim}{\mu}_{\ln X} + \underset{\approx}{L}\underset{\sim}{Z} \qquad [6]$$

where $\underset{\sim}{x}_i$ is a point in the field, $\underset{\approx}{L}$ is a lower triangular matrix satisfying $\underset{\approx}{L}\underset{\approx}{L}^T = \underset{\approx}{C}$ (obtained using a Cholesky Decomposition), and $\underset{\sim}{Z}$ is a vector of n_p independent standard normal random variables (mean zero, unit variance).

The lognormal random field, $\underset{\sim}{X}$, is obtained from the normal field, $\underset{\sim}{G}$, using the following transformation:

$$\underset{\sim}{X} = \exp\{\underset{\sim}{G}\} \qquad [7]$$

CMD is simple yet inefficient for large fields, as a field of size $n \times n$ requires a covariance matrix of size $n^2 \times n^2$. However, CMD is known to be suitable for small random fields, similar to the ones used in this research with the maximum field size of 4×4 as depicted in Figure 1. The CMD method is discussed in detail in Fenton and Griffiths (2008).

4 DESIGN APPROACH

Both load and resistance are assumed to be lognormally distributed. An individual pile is initially subjected to individual load F_i having mean μ_{F_i} and the first step here is to determine the required mean design pile resistance, μ_R, using reliability-based design concepts. That is, the pile is to be designed to successfully support the initial individual load F_i with some target reliability index, β_i, i.e.

$$\begin{aligned}\mathrm{P}[R_i > F_i] &= \mathrm{P}[R_i / F_i > 1] = \mathrm{P}[\ln(R_i / F_i) > 0] \\ &= \mathrm{P}[\ln W_i > 0] = \Phi\left(\frac{\mu_{\ln W}}{\sigma_{\ln W}}\right) = \Phi(\beta_i)\end{aligned} \qquad [8]$$

where Φ is the standard normal cumulative distribution function, and $\beta_i = \mu_{\ln W} / \sigma_{\ln W}$ is the reliability index for an individual pile. In Eq. 8, the quantity W_i is defined as the ratio of resistance over load, and as such, is random and also follows a lognormal distribution. Thus,

$$\ln W_i = \ln\left(\frac{R_i}{F_i}\right) = \ln R_i - \ln F_i \qquad [9]$$

is normal with parameters:

$$\begin{aligned}\mu_{\ln W} &= \mu_{\ln R_i} - \mu_{\ln F_i} \\ \sigma^2_{\ln W} &= \sigma^2_{\ln R_i} + \sigma^2_{\ln F_i}\end{aligned} \qquad [10]$$

where independence between the random variables R_i and F_i (or $\ln R_i$ and $\ln F_i$) was assumed to compute $\sigma^2_{\ln W}$. With reference to Eq. 4, the mean and variance of resistance, R_i, are

$$\begin{aligned}\sigma^2_{\ln R_i} &= \ln(1+v_R^2) \\ \mu_{\ln_{R_i}} &= \ln(\mu_{R_i}) - \frac{1}{2}\sigma^2_{\ln R_i}\end{aligned} \qquad [11]$$

Similarly,

$$\begin{aligned}\sigma^2_{\ln F_i} &= \ln(1+v_{F_i}^2) = \ln(1+v_T^2) = \sigma^2_{\ln F_T} \\ \mu_{\ln F_i} &= \ln(\mu_{F_i}) - \frac{1}{2}\sigma^2_{\ln F_i} = \ln(\mu_T / n_p) - \frac{1}{2}\sigma^2_{\ln F_T}\end{aligned} \qquad [12]$$

Substituting Eq.'s 11 and 12 into Eq. 10 gives

$$\sigma^2_{\ln W} = \ln(1+v_R^2) + \ln(1+v_T^2) = \ln\left[(1+v_R^2)(1+v_T^2)\right]$$

$$\begin{aligned}\mu_{\ln W} &= \ln(\mu_R) - \frac{1}{2}\sigma^2_{\ln R} - \ln(\mu_{F_i}) + \frac{1}{2}\sigma^2_{\ln F_i} \\ &= \ln\left(\mu_R / \mu_{F_i}\right) - \ln\left(\frac{1+v_R^2}{1+v_T^2}\right)^{1/2} \\ &= \ln\left(\frac{\mu_R / \mu_{F_i}}{\sqrt{\left(1+v_R^2\right)/\left(1+v_T^2\right)}}\right)\end{aligned} \qquad [13]$$

The individual reliability index is obtained using Eq. 13 as follows,

$$\beta_i = \frac{\mu_{\ln W}}{\sigma_{\ln W}} = \frac{\ln\left(\frac{\mu_R / \mu_{F_i}}{\sqrt{\left(1+v_R^2\right)/\left(1+v_T^2\right)}}\right)}{\sqrt{\ln\left[(1+v_R^2)(1+v_T^2)\right]}} \qquad [14]$$

Solving Eq. 14 for μ_R gives

$$\begin{aligned}\mu_R &= \mu_{F_i} \exp\left(\beta_i \sqrt{\ln\left[(1+v_R^2)(1+v_T^2)\right]}\right)\sqrt{\left(1+v_R^2\right)/\left(1+v_T^2\right)} \\ &= \mu_{F_i} \exp\left(\beta_i \sigma_{\ln W}\right)\sqrt{\left(1+v_R^2\right)/\left(1+v_T^2\right)}\end{aligned} \qquad [15]$$

which indicates that the design mean resistance, μ_R, depends on mean individual load, μ_{F_i}, individual target reliability index, β_i, as well as load and resistance coefficients of variation, v_T and v_R, respectively.

Once a design mean resistance, μ_R, is obtained via Eq. 15, the simulation process is carried out as follows:

1. Two lognormal random fields, each of size n_p, are generated representing loads, F_i, and resistances, R_i, associated with individual piles in a pile system arranged as depicted in Figure 1.
2. Individual piles are ranked from those with the smallest R_i / F_i ratio to those with the largest ratio.
3. The system survives if all R_i / F_i ratios exceed 1. If the first m piles (after ranking above) have R_i / F_i ratios less than 1, it means that these piles have been overloaded and they cannot support their full applied load F_i . In this case, the residual load which is not carried by these m piles must be distributed to the remaining $n_p - m$ piles according to Eq's 1 and 2.
4. Steps 2 and 3 are repeated for the remaining "unfailed" piles until the success (or failure) of the pile system is decided.

The above process is repeated n_{sim} times after which the system failure probability is estimated using

$$p_f = n_f / n_{sim} \qquad [16]$$

where n_f is the number of realizations resulting in a system failure, and n_{sim} is the total number of realizations.

5 RESULTS AND DISCUSSION

The objective of this section is to investigate how the individual reliability, β_i, relates to system reliability β_{sys}. The parameters used in this case study are listed in Table 1.

Table 1. Input parameters used in simulation

Parameters	Values Considered
n_p	4, 9, 16
μ_T	100 KN
v_T	0.1
v_R	0.15
$\theta_F / \mathrm{s} = \theta_R / \mathrm{s}$	1
n_{sim}	100,000+
a	0, 0.5, 1.0

The simulation involves at least $n_{sim} = 100,000$ realizations, increasing as the failure probability decreases (β_{sys} increases). At the minimum, the standard deviation of the failure probability estimate is $\sqrt{p_f(1-p_f)/n_{sim}} \simeq 0.003\sqrt{p_f}$ for small failure probability p_f. This means that if $p_f = 1\times10^{-4}$, then the standard deviation of its estimate is about 3×10^{-5} and therefore, $n_{sim} = 100,000$ can reasonably resolve probabilities down to about 10^{-4}. For larger reliability indices, the number of simulations required is computed as $n_{sim} \approx 96/\Phi\left(-\beta_{sys}\right)$ which corresponds to a 95% confidence that the relative error on the estimated failure probability is no more than 20%.

Based on results presented in Fenton et al. (2015), the worst case correlation length is approximately equal to the pile spacing, s, and hence $\theta_F = \theta_R = s$ is used in the generation of random fields in this study, which is believed to yield conservative estimates of β_i for given target system reliability β_{sys} .

Figures 2-4 illustrate how the reliability of individual piles, β_i, relates to the system reliability, β_{sys}. The task now is to determine β_i from Eq. 15 for given β_{sys}. This is accomplished through the following steps;

1. Initially guess that $\beta_i = \beta_{sys}$,
2. Compute the required design value of μ_R using Eq. 15,
3. Estimate probability of foundation system failure, p_f , according to the algorithm given in the previous section and Eq. 16,

4. Compute $\beta'_{sys} = \Phi^{-1}(1 - p_f)$,
5. If $\beta'_{sys} > \beta_{sys}$ then reduce β_i and repeat steps 2 to 5 until $\beta'_{sys} = \beta_{sys}$.

Figures 2-4 can be used for design by drawing a vertical line at the target system reliability index, β_{sys}, and then reading off the required β_i for a given n_p. For example, for a foundation system consisting of $n_p = 9$ piles, and a moderate target system reliability of $\beta_{sys} = 3.0$, corresponding to $p_f \simeq 1/1000$, the recommended single pile reliability index is given by Figure 2 to be $\beta_i = 1.2$ for $a = 0$. When $a = 0$ in Eq. 1, the pile resistance is assumed to never be less than R_i, even if the pile "fails" ($F_i > R_i$). At the other extreme, where the pile resistance is assumed to go to zero as soon as it fails, i.e., $a = 1$ in Eq. 1, then Figure 4 recommends that $\beta_i = 2.9$ should be used in the design of an individual pile.

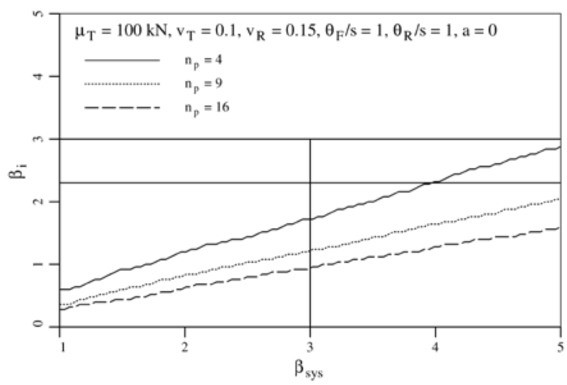

Figure 2. Plot of β_i versus β_{sys} for $\mu_T = 100$ kN, $v_T = 0.1$, $v_R = 0.15$, $a = 0$, and various number of piles, n_p, according to simulation results.

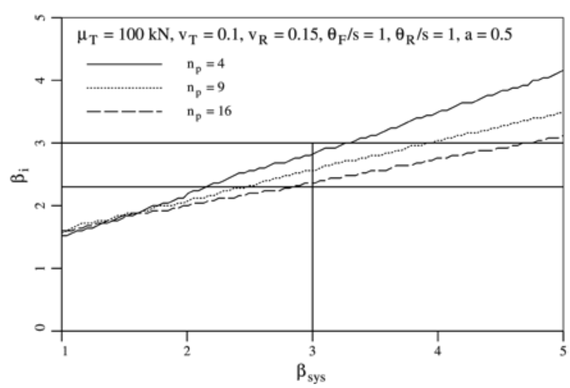

Figure 3. Plot of β_i versus β_{sys} for $\mu_T = 100$ kN, $v_T = 0.1$, $v_R = 0.15$, $a = 0.5$, and various number of piles, n_p, according to simulation results.

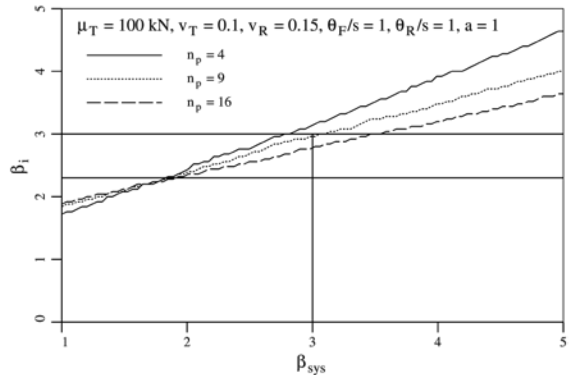

Figure 4. Plot of β_i versus β_{sys} for $\mu_T = 100$ kN, $v_T = 0.1$, $v_R = 0.15$, $a = 1$, and various number of piles, n_p, according to simulation results.

Generally speaking, a reliability index of β_i = 3.0 ($p_f \simeq 1/1000$) is prescribed in geotechnical design practice as the target reliability index of an individual pile in non-redundant pile systems ($n_p \le 4$), and $\beta_i = 2.3$ ($p_f \simeq 1/100$) for redundant pile systems ($n_p \ge 5$) (Zhang et al. 2001, Paikowsky et al., NCHRP, 2004, Allen 2005, and Barker et al., NCHRP, 1991). According to Zhang et al. (2001), a β_{sys} value of 3.0 requires a β_i = 2.0 to 2.8 for redundant systems, which is in agreement with Figure 3 results generated for $a = 0.5$. In other words, taking into account about half of the resistance of the failed piles, will require individual reliability index of $\beta_i = 2.3 - 2.5$ in order to achieve a target system reliability of $\beta_{sys} = 3.0$ for redundant systems($n_p \ge 5$).

6 CONCLUSIONS

In this paper, the reliability of a pile system for various levels of pile redundancy and resistance statistics, are studied, and a relationship between reliability of a pile system and the reliability of its individual components is established via simulation. The simulation considers correlation between pile resistances (through the soil) and between loads (through the structural stiffness) and the impact of these on the reliability of the pile system. The results of this paper are plots showing individual reliability index as a function of the system reliability index and the number of piles. These plots can be used to determine the required reliability index to use in the design of individual piles when the system reliability index is known.

ACKNOWLEDGEMENTS

The authors are thankful for the support provided by the Natural Sciences and Engineering Research Council of Canada and by a Special Research Project Grant from the Ministry of Transportation of Ontario.

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