STATISTICS OF FREE SURFACE FLOW THROUGH STOCHASTIC EARTH DAM

By Gordon A. Fenton,[1] and D. V. Griffiths,[2] Members, ASCE

ABSTRACT: Even though soil is a highly variable material, the analysis of flow through earth dams typically proceeds deterministically and results can sometimes be quite misleading. In fact it is well known that soil permeability varies randomly from point to point in space and an improved earth dam model should incorporate this variability. In this paper the soil permeability in an earth dam of variable geometry is viewed as a weakly stationary spatially random field following a lognormal distribution with prescribed mean, variance, and spatial correlation structure. The mean and variance of the total flow rate through the dam and free surface drawdown are estimated using Monte Carlo simulations. A simplified empirical approach to the prediction of the mean and variance of the flow rate is presented to allow these quantities to be approximated easily and without resorting to simulation.

INTRODUCTION

Many water retaining structures in North America are earth dams and the prediction of flow through such structures is of interest to planners and designers. Although it is well known that soils exhibit highly variable hydraulic properties, the prediction of flow rates through earth dams is generally performed using deterministic models. This paper introduces a second-order stationary (or weakly stationary) random model of an earth dam and investigates the effects of spatially random hydraulic properties on two quantities of classical interest: (1) The total flow rate through the dam; and (2) the amount of drawdown of the free surface at the downstream face of the dam. The drawdown is defined as the elevation of the point on the downstream face of the dam at which the water first reaches the dam surface. Other issues which relate more to the structural reliability of an earth dam, such as failure by piping and flow along eroding fractures, are not addressed in the current study. Work in this area by the authors is ongoing. Here it is assumed that the permeability field is representable by a mean-square continuous random field and that interest is in the stable, steady state, flow behavior. This study contributes to the understanding of the stochastic behavior of flows through soils, a classical problem in geomechanics.

The computation of flow through an earth dam is complicated by the fact that the location and profile of the free surface is not known a priori and must be determined iteratively. Nevertheless, the finite element code required to perform such an analysis is really quite straightforward, involving a simple Darcy flow model and iteratively adjusting the nodal elevations along the free surface to match their predicted potential heads (Smith and Griffiths 1988). Lacy and Prevost (1987), among others, suggest an alternative approach which employs a fixed mesh but the approach given by Smith and Griffiths was selected here due to its simplicity.

When the permeability is viewed as a spatially random field, the equations governing the flow become stochastic. The random field characterizes uncertainty about the permeability at all points in the dam and from dam to dam. The flow through the dam will thus also be uncertain and this uncertainty can be expressed by considering the probability distribution of the total flow rate. In turn, this distribution can be used to calculate probabilities related to flow.

Stochastic problems of this nature can sometimes be solved using stochastic finite element techniques [for a good discussion of the method, see Vanmarcke et al. (1986) and Vanmarcke (1994)]. These methods are based on first or second order perturbation approaches in which the random parameters are decomposed into a mean (deterministic) part and a random fluctuation. Unfortunately, the resulting relationships are derived on the assumption that the fluctuating part is small compared to the mean part with coefficients of variation less than about 20%. Since soil permeability often shows much higher variability, in orders of magnitude, the stochastic finite element techniques are prone to considerable error and cannot be used for many flow problems of interest.

Under certain conditions, analytical results are available which can be used to characterize flow through a random medium. Indelman and Abramovich (1994), Dagan (1993), Gelhar (1993), and Dykaar and Kitanidis (1992), among others, all develop techniques of estimating an effective permeability that can be used to predict the mean flow. These results are obtained under the assumption that the domain is of infinite size and the mean flow is unidirectional (through a spectral decomposition algorithm, Dykaar and Kitanidis relieve both of these restrictions but still only provide a method of finding the mean flow). In many cases of interest to practicing geotechnical engineers, the domain is not infinite, the flow paths are not unidirectional on average, and the total flow rate will show considerable variability from realization to realization. In fact estimation of the flow variance is an essential ingredient in any probability estimate and flow variability will arise in most finite domains when modeled randomly.

It is instructive at this point to review issues relating to the ergodic hypothesis and discuss its implications on the results presented here. It is not uncommon that statistics of the permeability field (mean, variance, correlation structure) must be estimated from a single realization, in this case from a single earth dam, since nature may not provide an ensemble of independent realizations coming from the same governing distribution. If this is the case, the ergodic hypothesis must be assumed, inplying that all possible states of the random process occur in the single realization. This allows accurate estimates of the statistics by averaging over space rather than over an ensemble of realizations. In turn, ergodicity implies that the permeabilities from point to point become effectively independent beyond distances significantly smaller than the size of the sampling region. Thus the assumption of ergodicity states something about the scale over which permeabilities are effectively correlated, namely that this scale is small compared

[1]Assoc. Prof., Dept. of Appl. Math., Tech. Univ. of Nova Scotia, Halifax, Nova Scotia, B3J 2X4 Canada.

[2]Prof., Geomech. Res. Ctr., Colorado School of Mines, Golden, CO 80401-1887.

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to the sampling domain. In the sequel, this scale will be referred to as the scale of fluctuation and its mathematical definition will appear later.

Under the ergodic hypothesis, statistical inference proceeds by collecting data at various spatial locations and estimating the mean, variance, and spatial correlation structure by averaging over the spatial observations. It is generally difficult to verify the assumption of ergodicity once it is made. Typical approaches involve estimating the scale of fluctuation and checking that it is indeed significantly smaller than the sampling domain (Gelhar 1993; Dagan 1989). Problems arise, though, when the scale of fluctuation governing the process is of the same order or greater than the sampling domain. Such processes appear to be non-stationary with trends in the mean and variance as functions of space. The usual cure is to functionally account for these trends and estimate the statistics of the residual process, which is now guaranteed to be second-order stationary (and invariably yields an estimated scale of fluctuation considerably smaller than the sampling domain). The only way to clearly establish if these are in fact nonstationary trends or simply a result of looking at a limited segment of a process with a large scale of fluctuation is to have available other realizations of the process. Another danger is that what appears to be a stationary random process after sampling, for which the ergodic assumption appears to hold, may in fact be part of a process with many different scales of behavior. An example of such a process might be found in the weather; when observed over only a month, the process might appear stationary and ergodic. However, it is well known that correlation scales for such processes span months, years, and even thousands of years. If one looks at the depositional processes leading to soil formation on the surface of the earth, it is easy to imagine that many different correlation scales can be also found in soils.

As will be shown later, earth dams for which the ergodic hypothesis holds show little variability in total flow rate and can be analyzed using the effective permeabilities discussed above. This is by virtue of the fact that the scale of fluctuation is small relative to the flow regime. In this paper, because of the observations made previously, the ergodic restriction is relieved and simulation results are produced for both ergodic (small scale of fluctuation) and nonergodic (large scale of fluctuation) earth dams. This allows also the use of statistics estimated from a variety (ensemble) of similar earth dams for the purposes of predicting flow rate probabilities in dams not yet built or for which field information is absent. If field information exists for the dam of interest, then a probabilistic analysis of the dam would typically proceed using a conditioned random field that explicitly accounts for the observed data. This is beyond the scope of the current paper.

In cases where the soil properties show strong variability and the domain is not significantly larger than the scale of fluctuation, the only alternative currently is to perform Monte Carlo simulations, with a suitable number of realizations, in order to produce accurate estimates of the flow rate statistics (assuming accurate input parameters). Although considered by some to be a brute force approach, Monte Carlo simulations nevertheless provide a simple means of estimating the full distribution of response quantities of interest which could not otherwise be obtained. From these distributions, simple, albeit empirical, relationships can be derived to aid the designer in estimating the distribution parameters directly from the parameters of the random field model.

Following this reasoning, a Monte Carlo analysis approach has been adopted herein. A sequence of 1,000 realizations of spatially varying soil properties with prescribed mean, variance, and spatial correlation structure are generated and then analyzed separately to obtain a sequence of flow rates and free surface profiles. The mean and variance of the flow rate and drawdown statistics can then be estimated directly from the sequence of computed results. The number of realizations was selected so that the variance estimator of the logarithm of total flow rate had coefficient of variation less than 5% (computed analytically under the assumption that log-flow rate is normally distributed). Note that since these statistics are estimated from an ensemble (of simulations), there is no reason to restrict the analysis to the ergodic case.

Because the analysis is Monte Carlo in nature, the results are strictly only applicable to the particular earth dam geometries and boundary conditions studied, however the general trends and observations may be extended to a range of earth dam boundary value problems. An empirical approach to the estimation of flow rate statistics and governing distribution is presented to allow these statistics to be easily approximated, that is without the necessity of the full Monte Carlo analysis. This simplified procedure needs only a single finite element analysis and knowledge of the variance reduction due to local averaging over the flow regime and will be discussed in detail in the section of the present paper entitled "Empirical Estimation of Flow Rate Statistics."

Fig. 1 illustrates the earth dam geometries considered in this study, each shown for a realization of the soil permeability field. The square and rectangular dams were included since these are classical representations of the free surface problem (Dupuit problem). The other two geometries are somewhat more realistic. The steep sloped dam labeled Dam 1 in Fig. 1 can be thought of as a clay core held to its shape by highly permeable back fill having negligible influence on the flow rate (and thus the fill is not explicitly represented).

Fig. 2 shows two possible realizations of Dam 1. It can be seen that the free surface typically lies some distance below the top of the dam. Because the position of the surface is not known a priori, the flow analysis necessarily proceeds iteratively. Under the free surface, flow is assumed to be governed by Darcy's law characterized by an isotropic permeability, $K(\underset{\sim}{x})$, where $\underset{\sim}{x}$ = spatial location

$$\nabla \cdot \underset{\sim}{q} = 0; \quad \underset{\sim}{q} = -K(\underset{\sim}{x})\nabla\phi \tag{1}$$

where $\underset{\sim}{q}$ = specific discharge vector; and ϕ = hydraulic head.

RANDOM FIELD PERMEABILITY MODEL

The permeability, $K(\underset{\sim}{x})$, is assumed to follow a lognormal distribution, consistent with the findings of Freeze (1975), Hoeksema and Kitanidis (1985), and Sudicky (1986) and with the work of Griffiths and Fenton (1993), with mean μ_k and variance σ_k^2. Thus $\ln K$ is normally distributed (Gaussian) with mean $\mu_{\ln k}$ and variance $\sigma_{\ln k}^2$, where

$$\sigma_{\ln k}^2 = \ln\left(1 + \frac{\sigma_k^2}{\mu_k^2}\right) \tag{2a}$$

$$\mu_{\ln k} = \ln(\mu_k) - \frac{1}{2}\sigma_{\ln k}^2 \tag{2b}$$

Since $K(\underset{\sim}{x})$ is a spatially varying random field, there will also be a degree of correlation between $K(\underset{\sim}{x})$ and $K(\underset{\sim}{x}')$, where $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ = any two points in field. Intuitively it makes sense that the permeability at $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ will be quite similar if $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are close together. Alternatively, if the two points are widely separated less correlation may be expected. Mathematically this concept is captured through the use of a spatial correlation function, which, in the present study, is an exponentially decaying function of separation distance $\underset{\sim}{\tau} = \underset{\sim}{x} - \underset{\sim}{x}'$ (Sudicky 1986)

$$\rho(\underset{\sim}{\tau}) = e^{-2|\underset{\sim}{\tau}|/\theta_k} \tag{3}$$

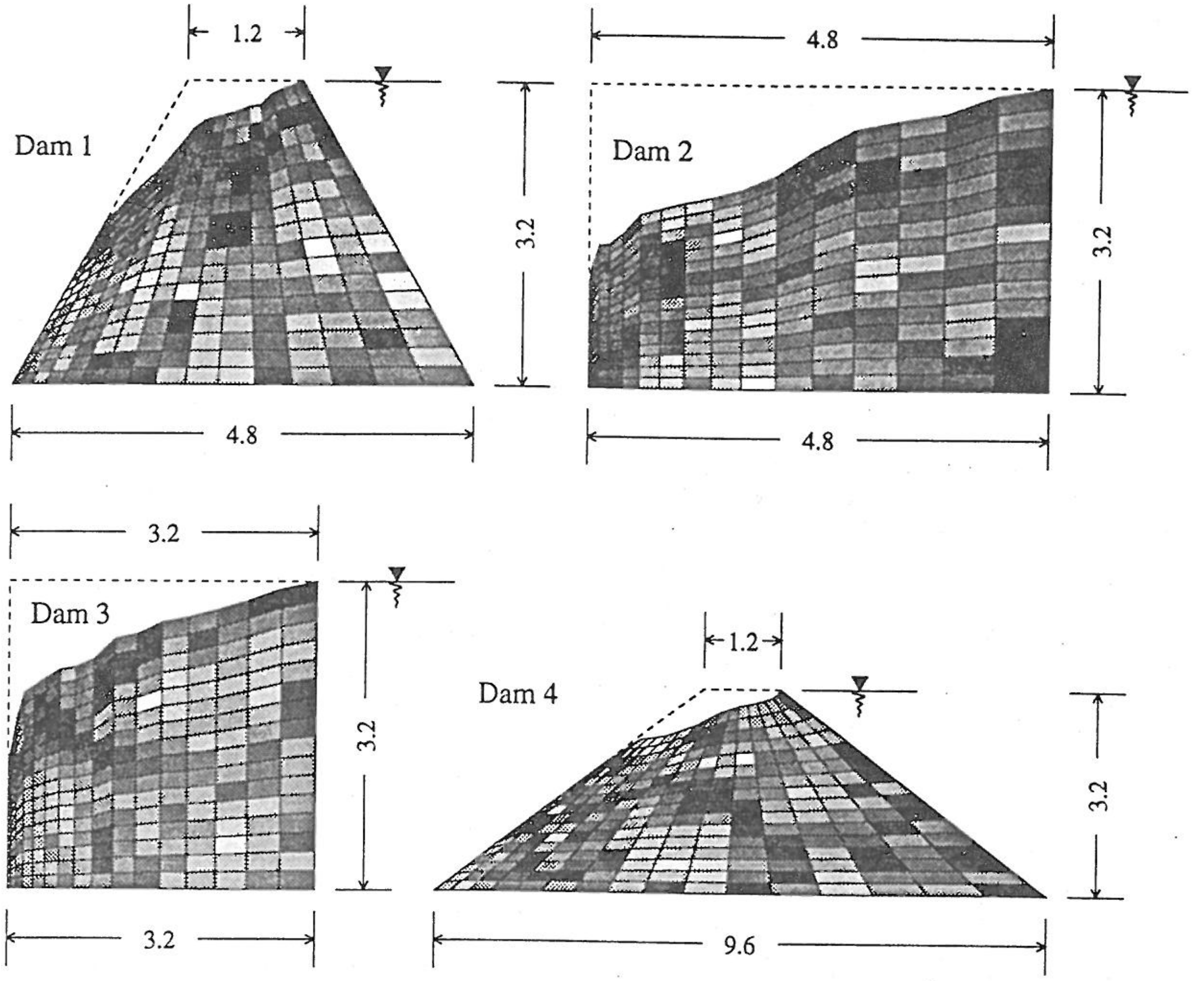

FIG. 1. Earth Dam Geometries Considered in Study

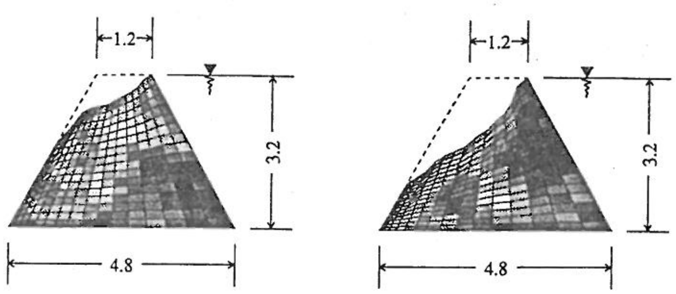

FIG. 2. FEM Discretization of Dam 1 Shown in Fig. 1; Two Possible Realizations

where θ_k = scale of fluctuation [also referred to as integral scale; θ_k is twice the integral scale defined by Dagan (1989)] defined in terms of the correlation function as

$$\theta_k = \int_{-\infty}^{\infty} \rho(\tau)\, d\tau \quad (4)$$

where in the isotropic case $\tau = |\underline{\tau}|$. Loosely speaking, the scale of fluctuation can be thought of as the distance over which points on the random field show substantial correlation. Eq. (3) is a Gauss-Markov model for which short scales of fluctuation yield "rougher" or less correlated fields, while longer scales yield smoother or more correlated fields. When $\theta_k \to \infty$, the random field becomes completely correlated; all points on the field have the same permeability, assuming stationary mean, and variation only occurs from realization to realization (the permeability is still random, just not within a specific realization). When $\theta_k = 0$ the permeability at all points become independent, a physically unrealizable situation. In fact scales of fluctuation less than the size of laboratory samples used to estimate permeability have little meaning since Darcy's law is a continuum representation where the permeability is measured at the laboratory scale. In this light, the use of the phrase "permeability at a point" herein really means permeability in a representative volume (of the laboratory scale) centered at the point.

As discussed previously, when the scale of fluctuation is of the same size or greater than the dimensions of the dam, realizations of the random field may exhibit what appears to be nonstationary behavior over the observed domain (evidenced by trends in the mean and covariance structure); the random field itself is still second-order stationary, but it is essentially being viewed over a segment too small to illustrate this stationarity. If similar trends exist in earth dams which have been field sampled, there are two ways of accommodating them: (1) Account for the trends functionally and randomly model only the stationary residual (which ensures estimated scales of fluctuation significantly smaller than the domain size); or (2) assume that the earth dam is just a segment of a much larger mass of soil and attempt to estimate its scale of fluctuation directly. Typically the latter approach depends on more information than available from a single earth dam, but it has the advantage of not prescribing a functional form for the trends in the permeability field (if these are not known for the earth dam being studied).

Another possibility which has been receiving considerable attention in the last few years and which holds much promise is to use multiscale or infinite-scale random field models such as statistically self-similar or fractal processes, characterized by a fractal dimension, to represent the random permeability field. These processes exhibit apparently nonstationary behavior at all finite observation scales and have the advantage of not depending on a single scale of fluctuation which may be difficult to estimate from a single realization especially in the presence of apparently nonstationary trends. While there is evidence that statistically self-similar processes are representative of a large range of physical phenomena [see, for example, Mandelbrot (1982)], most geostatistical effort has gone into developing exponentially decaying correlation models characterized by a single scale of fluctuation (possibly in each coordinate direction). For this reason statistically self-similar models are not pursued further in the current study.

Simulations of the soil permeability field proceeds in two steps; first an underlying Gaussian random field, $G(\underset{\sim}{x})$, is generated with mean zero, unit variance, and spatial correlation function (3) using the Local Average Subdivision (LAS) method introduced by Fenton and Vanmarcke (1990). The LAS algorithm produces a regular grid of elements, each element having a value assigned to it equal to the local average of the enclosed random field. This approach is well suited to use with finite-element analyses since local average values can be easily mapped to the finite elements and element statistics are properly related to the element size [as the element size increases, the local average variance decreases due to the averaging effect; see Vanmarcke (1984)].

Next, since the permeability is assumed to be lognormally distributed, values of K_i, where i denotes the ith element, are obtained through the transformation

$$K_i = \exp[\mu_{\ln k} + \sigma_{\ln k} G(\underset{\sim}{x}_i)] \quad (5)$$

where $\underset{\sim}{x}_i$ is the centroid of the ith element and $G(\underset{\sim}{x}_i)$ is the local average value generated by the LAS algorithm of the cell within which $\underset{\sim}{x}_i$ falls. As will be discussed later, the finite-element mesh is deformed while iterating to find the free surface so that local average elements only approximately match the finite elements in area. Thus for a given realization, the spatially "fixed" permeability field values are assigned to individual elements according to where the element is located on each free-surface iteration.

Both permeability and scale of fluctuation are assumed to be isotropic in the present study. Although layered construction of an earth dam may lead to some anisotropy relating to the scale of fluctuation and permeability, this is not thought to be a major feature of the reconstituted soils typically used in earth dams (unfortunately, to the writers' knowledge, there is no evidence to formally support or refute such a statement). In contrast, however, natural soil deposits can exhibit quite distinct layering and stratification in which anisotropy can not be ignored. Note that random fields with ellipsoidally anisotropic correlation functions, for example of the form

$$\rho(\underset{\sim}{\tau}) = \exp\left(-2\sqrt{\frac{\tau_1^2}{\theta_1^2} + \frac{\tau_2^2}{\theta_2^2}}\right) = \exp\left(-\frac{2}{\theta_1}\sqrt{\tau_1^2 + \frac{\tau_2^2\theta_1^2}{\theta_2^2}}\right) \quad (6)$$

where θ_1 and θ_2 = directional scales of fluctuation, can always be transformed into isotropic forms by suitably scaling the coordinate axes. In this example by using $x_2' = x_2(\theta_1/\theta_2)$, where x_2 = space coordinate measured in same direction as τ_2, (6) becomes isotropic with scale θ_1 and lag $\tau = \sqrt{\tau_1^2 + (\tau_2')^2}$, with τ_2' measured with respect to x_2'. Thus, if anisotropy is significant, such a transformation can be performed to allow the use of the results presented here, bearing in mind that it is the transformed geometry which must be used in the sequel.

The model itself is two-dimensional, which is equivalent to assuming that the stream-lines remain in the plane of analysis. This will occur if the dam ends are impervious and if the scale of fluctuation in the out-of-plane direction is infinite (implying that soil properties are constant in the out-of-plane direction). Clearly the latter condition will be false; however a full three-dimensional analysis is beyond the scope of the present study. It is believed that the two-dimensional analysis will still yield valuable insights.

FINITE-ELEMENT MODEL

For a given permeability field realization, the free surface location and flow through the earth dam is computed using a two-dimensional iterative finite element model derived from Smith and Griffiths (1988), Program 7.1. The elements are 4-node quadrilaterals and the mesh is deformed on each iteration until the total head along the free surface approaches its elevation head above a pre-defined horizontal datum. Convergence is obtained when the maximum relative change in the free surface elevation at the surface nodes becomes less than 0.005. Fig. 2 illustrates two possible free surface profiles corresponding to different permeability field realizations with the same input statistics.

When the downstream face of the dam is inclined, the free surface tends to become tangent to the face resulting in finite elements that can be severely skewed, leading in turn to inaccurate numerical results. This difficulty is overcome by proportionately shifting the mesh as the free surface descends to get a finer mesh near the top of the downstream face. Because of the mesh deformation taking place in each iteration along with the need to maintain the permeability realization as spatially fixed, the permeabilities assigned to each element are obtained by mapping the element centroids to the permeability field using (5). Thus the local average properties of the random field are only approximately reflected in the final mesh; some of the smaller elements may share the same permeability if adjacent elements fit inside a cell of the random field. This is believed to be an acceptable approximation, leading to only minor errors in the overall stochastic response of the system, as discussed next.

Preliminary tests performed for the present study indicated that the response statistics only began to show significant deviations when fewer than 5 elements were used in each of the two coordinate directions. In the current model 16 elements were used in each direction (256 elements in total). This ensures reasonable accuracy even in the event that some elements are mapped to the same random field element. Because the elements are changing size during the iterative process, implying that the local average properties of the random field generator are only approximately preserved in the final mesh, there is little advantage to selecting a local average random field generator over a point process generator such as the fast Fourier transform (FFT) or Turning Bands (TBM) methods. The LAS algorithm was selected for use here primarily because it avoids the possible presence of artifacts (in the form of streaks) in individual realizations arising in TBM realizations and the symmetric covariance structure inherent in the FFT algorithm (Fenton 1994). The LAS method is also much easier to use than the FFT approach.

Flow rate and drawdown statistics for the earth dam are evaluated over a range of the statistical parameters of K. Specifically the mean and standard deviation of the total flow rate, m_Q and s_Q, and the drawdown, m_Y and s_Y, are estimated for σ_k/μ_k = {0.1, 0.5, 1.0, 2.0, 4.0, 8.0} and θ_k = {0.1, 0.5, 1.0, 2.0, 4.0, 8.0} by averaging over 1,000 realizations for each (resulting in 6 × 6 × 1,000 = 36,000 realizations in total for each dam considered). An additional run using θ_k = 16 was performed for Dam 1 to verify trends at large scales of fluctuation. The mean permeability, μ_k, is held fixed at 1.0. The drawdown elevations Y are normalized by expressing them as a fraction of the overall (original) dam height.

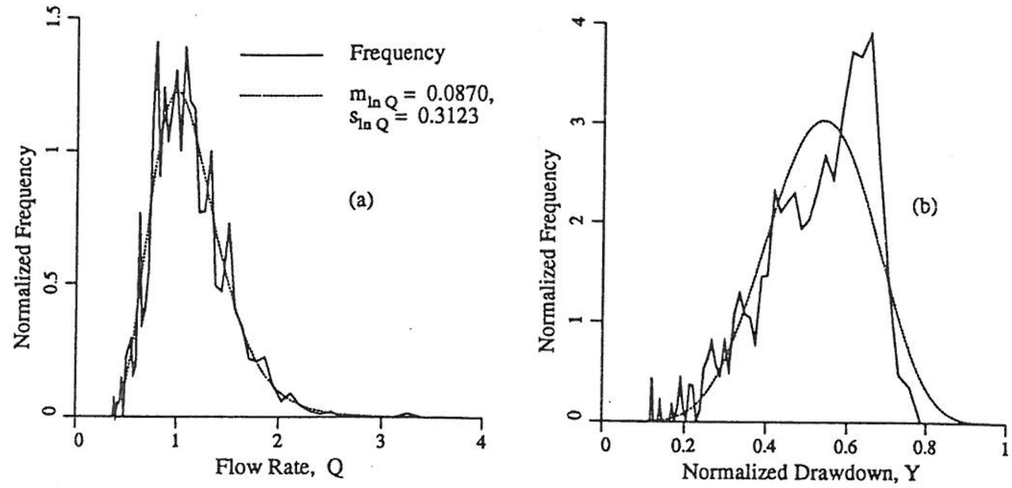

FIG. 3. Normalized Frequency Histograms of the Following: (a) Flow Rate; (b) Normalized Drawdown for Dam 1 with $\sigma_k/\mu_k = 1$ and $\theta_k = 1$

SIMULATION RESULTS

On the basis of 1,000 realizations, a normalized frequency histogram of flow rates and drawdowns can be produced for each set of parameters of $K(\underline{x})$. Typical histograms are shown in Fig. 3, with fitted lognormal and Beta distributions superimposed on the flow rate and normalized drawdown respectively. The parameters of the fitted distributions are estimated by the method of moments from the ensemble of realizations, which constitute a set of independent samples, using unbiased sample moments. For the lognormal distribution the estimators are

$$m_{\ln Q} = \frac{1}{n}\sum_{i=1}^{n} \ln Q_i \tag{7a}$$

$$s_{\ln Q}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\ln Q_i - m_{\ln Q})^2 \tag{7b}$$

where Q_i = total flow rate coming from the ith realization. For n = 1,000 realizations, the coefficient of variation of the estimators (assuming $\ln Q$ follows a normal distribution) $m_{\ln Q}$ and $s_{\ln Q}^2$ are $0.032\sigma_{\ln k}/\mu_{\ln k}$ and 0.045, respectively.

It can be seen that the lognormal distribution fits the flow rate histogram reasonably well, as is typical; 60% of the cases considered (based on 1,000 realizations each) satisfied the Chi-Square goodness-of-fit test at the 5% significance level. A review of the histograms corresponding to those cases not satisfying the test indicates that the lognormal distribution is still a reasonable approximation, but that the Chi-Square test is quite sensitive. For example, the histogram shown in Fig. 3(a) fails the Chi-Square test at all significance levels down to 0.15%. From the point of view of probability estimates associated with flow rates, it is deemed appropriate therefore to assume that flow rates are well approximated by the lognormal distribution and all subsequent statistics of flow rates are determined from the fitted lognormal distribution.

Since the normalized drawdown is bound between 0 and 1 it was felt that perhaps a Beta distribution might be an appropriate fit. Unfortunately the fit, obtained by method of moments using unbiased sample moments of the raw data, was typically poor; the histogram shown in Fig. 3(b) has sample mean and standard deviation 0.533 and 0.125, respectively, giving Beta distribution parameters α = 7.91 and β = 6.93. The fitted distribution fails to capture the skewness and upper tail behavior. Nevertheless, the drawdown mean and variance can be estimated reasonably accurately even though its actual distribution is unknown. For 1,000 realizations, the estimators of the mean and variance of normalized drawdown have coefficients of variation approximately $0.032 s_Y/m_Y$ and 0.045, using a normal distribution approximation.

The estimated mean and variance of the total log-flow rate, denoted here as $m_{\ln Q}$ and $s_{\ln Q}^2$ respectively, are shown in Fig. 4 as a function of the variance of log-permeability, $\sigma_{\ln k}^2 = \ln(1 + \sigma_k^2/\mu_k^2)$, and the scale of fluctuation, θ_k. These results are for Dam 1 and are obtained from (7). Clearly the mean log-flow rate tends to decrease from the deterministic value of $\ln(Q_{\mu_k}) = \ln(1.51) = 0.41$ (obtained by assuming $K = \mu_k = 1.0$ everywhere) as the permeability variance increases.

In terms of the actual flow rates which are assumed to be lognormally distributed, the transformations

$$m_Q = \exp(m_{\ln Q} + s_{\ln Q}^2/2) \tag{8a}$$

$$s_Q^2 = m_Q^2[\exp(s_{\ln Q}^2) - 1] \tag{8b}$$

can be used to produce the mean flow rate plot shown in Fig. 5. The apparent increase in variability of the estimators (see, for example, the θ_k = 16 case) is due in part to the reduced vertical range, but is also partly due to errors in the fit of the histogram to the lognormal distribution and the resulting differences between the raw data estimators and the log-data estimators.

It can be seen that the mean flow rate also reduces from the deterministic value, Q_{μ_k} = 1.51, with increasing $\sigma_{\ln k}^2$. The reduction is more pronounced for small scales of fluctuation but virtually disappears for scales of fluctuation considerably larger than the dam itself. It is known that as the scale of fluctuation becomes negligible compared to the size of the dam, the effective permeability approaches the geometric mean $K_G = \mu_k \exp\{-1/2\sigma_{\ln k}^2\}$ (Dagan 1989), which for fixed μ_k illustrates the reduction in flow rate. Intuitively, one can think of this reduction in mean flow rate by first considering one-dimensional flow down a pipe; the total flow rate down the pipe is heavily dependent on the minimum permeability encountered along the way. As the variance of the permeability increases, and in the case of small scales of fluctuation, the chances of getting a small permeability or blocked pipe also increases, resulting in a decreased mean flow rate. Similar, albeit less extreme, arguments can be made in the two-dimensional case, leading to the observed and predicted reduction in mean total flow rate as $\sigma_{\ln k}^2$ increases. As the scale of fluctuation increases to infinity, the mean flow rate, m_Q, becomes equal to Q_{μ_k}, independent of $\sigma_{\ln k}^2$, as illustrated by the θ_k = 16 case in Fig. 5. In this case, the random field is relatively uniform, and although individual realizations show considerable variability in total flow rate, the mean approaches the value predicted by $K = \mu_k$.

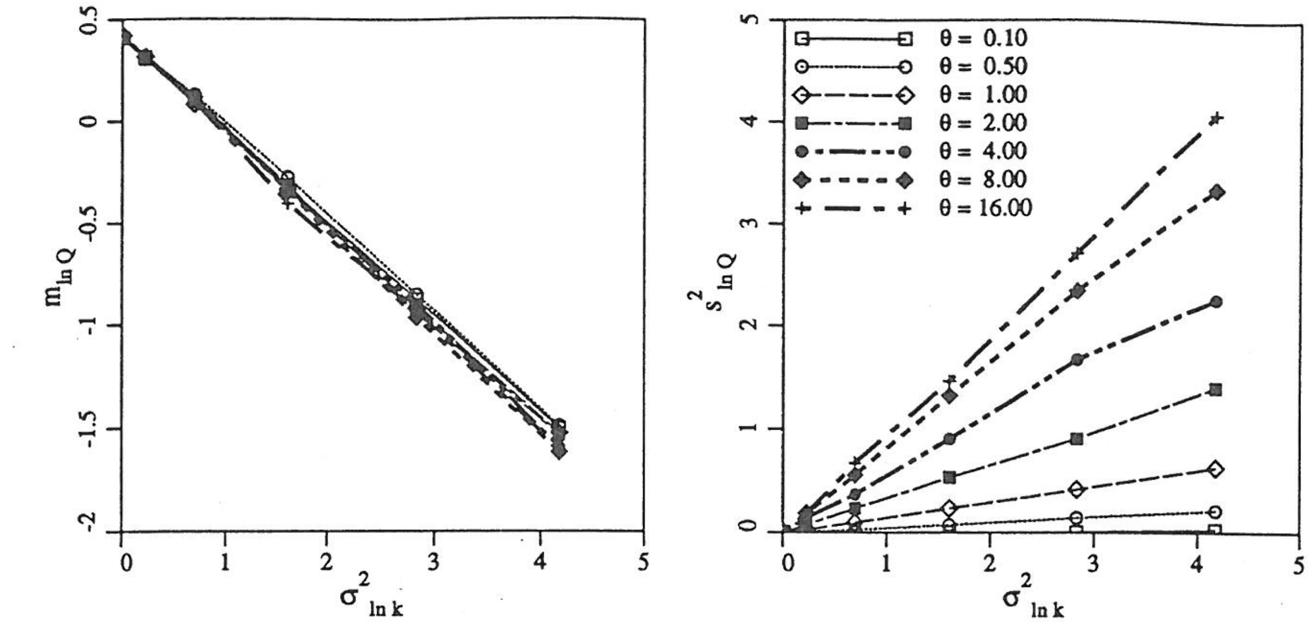

FIG. 4. Estimated Mean and Standard Deviation of Log-Flow Rate through Dam 1

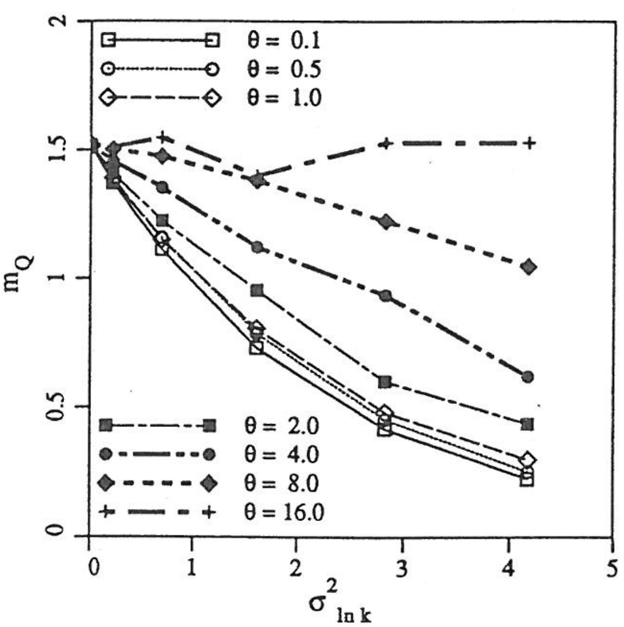

FIG. 5. Estimated Mean Flow Rate through Dam 1

For short scales of fluctuation, the variance of log-flow rate is small, as evidenced by $s_{\ln Q}$ in Fig. 4, increasing as the scale of fluctuation and $\sigma^2_{\ln k}$ increase. In the limit as $\theta_k \to \infty$, it can be shown that $\sigma^2_{\ln Q} = \sigma^2_{\ln k}$ and $\mu_{\ln Q} = \ln(Q_{\mu_k}) - \sigma^2_{\ln k}/2$, trends which are seen in Fig. 4 for $\theta_k = 16$. Similar results were found for the other dam geometries.

Fig. 6 shows the estimated mean and standard deviation of the normalized drawdown, m_Y and s_Y respectively, again for the earth Dam 1 shown in Fig. 1. It can be seen that although some clear patterns exist for the mean drawdown with respect to the scale of fluctuation and $\sigma^2_{\ln k}$, the magnitude of the mean drawdown is little affected by these parameters and remains close to $Y = 0.58$ of the total dam height obtained in the deterministic case with $K = \mu_k = 1.0$. Note that for $\theta_k = 4$, $\sigma^2_{\ln k} = 2.83$, the standard deviation of Y is estimated to be about 0.21, giving the standard deviation of the estimator m_Y to be about 0.0066. The 90% confidence interval on μ_Y is thus approximately [0.51, 0.53] for $m_Y = 0.52$. This observation easily explains the rather erratic behavior of m_Y observed in Fig. 6.

The variability of the drawdown, estimated by s_Y, is significantly affected by θ_k and $\sigma^2_{\ln k}$. For small scales of fluctuation relative to the dam size, the drawdown shows little variability even for high permeability variance. This suggests that, under these conditions, using a fixed free surface to model the dam may be acceptable. For larger scales of fluctuation, the drawdown shows more variability and the stochastic nature of the free surface location should be included in an accurate analysis. Although it may seem that the drawdown variability continues to increase with increasing scale of fluctuation, it is known that this is not the case. There will be a worst-case scale at which the drawdown variability is maximized; at even larger scales, the drawdown variability will decrease since in the limit as $\theta_k \to \infty$, the drawdown becomes equal to the deterministic result $Y = 0.58$, independent of the actual permeability. In other words, the drawdown becomes different from the deterministic result only in the presence of intermediate scale fluctuations in the permeability field. To investigate this phenomena, Dam 1 was analyzed for the additional scale $\theta_k = 16$ m, much greater than the earth dam dimension of around 3 to 4 m. It appears from Fig. 6 that the drawdown variance is maximized for $\theta_k = 4$ m, that is for θ_k of the order of the earth dam size.

EMPIRICAL ESTIMATION OF FLOW RATE STATISTICS

In that many designers and planners have neither the time nor the resources to perform full scale Monte Carlo simulations of flow through earth dams with spatially random properties, it is worthwhile investigating approximate or empirical methods of estimating the mean and variance of flow through an earth dam. In the following, a semiempirical approach is adopted, with the understanding that its accuracy in estimating flow statistics for problems other than those considered here is currently unknown. In practice the following results should be viewed as providing ball-park estimates, and more accurate estimates must currently be obtained via simulation.

The approach starts by noting that the mean, $\mu_{\ln Q}$, and variance, $\sigma^2_{\ln Q}$, of log-flow through a square two-dimensional domain with impervious top and bottom faces and constant head along both faces is accurately predicted by [on the basis of simulation studies, see Fenton and Griffiths (1993)]

$$\mu_{\ln Q} = \ln(Q_{\mu_k}) - \frac{1}{2}\sigma^2_{\ln k} \quad (9a)$$

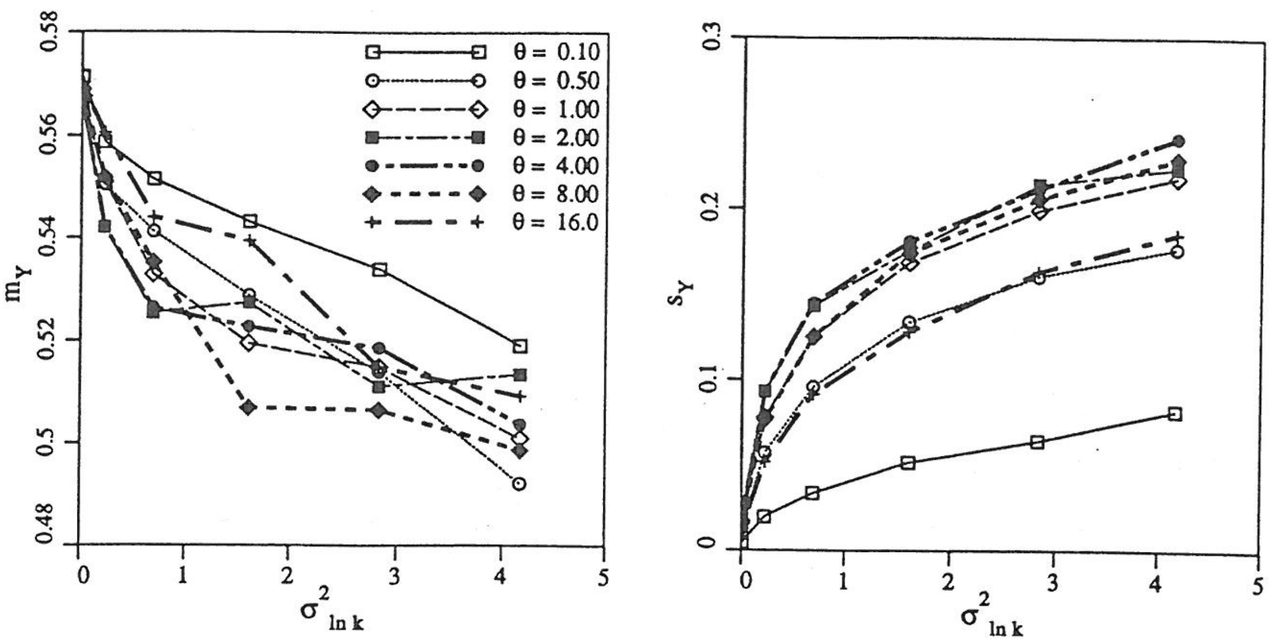

FIG. 6. Estimated Mean and Standard Deviation of Normalized Free Surface Drawdown for Dam 1

$$\sigma^2_{\ln Q} = \sigma^2_{\ln k}\gamma(D, D) \tag{9b}$$

with equivalent results in real space (assuming that Q follows a lognormal distribution) given by

$$\mu_Q = Q_{\mu_k} \exp\left\{-\frac{1}{2}\sigma^2_{\ln k}[1 - \gamma(D, D)]\right\} \tag{10a}$$

$$\sigma^2_Q = Q^2_{\mu_k} \exp\{-\sigma^2_{\ln k}[1 - \gamma(D, D)]\}\{\exp[\sigma^2_{\ln k}\gamma(D, D)] - 1\} \tag{10b}$$

where Q_{μ_k} = flow rate obtained in deterministic analysis of flow through domain having permeability $K(\underline{x}) = \mu_k$ everywhere; D = square root of domain area (i.e., side length); and $\sigma^2_{\ln k}\gamma(D, D)$ = variance of local average of random field $\ln(K)$ over domain $D \times D$. In the event that $D >> \theta_k$, so that $\gamma(D, D) \simeq 0$, 9(a) and 9(b) become equal to that predicted using the geometric mean of permeability, that is to the effective permeability defined by Dagan (1989) and Gelhar (1993). In the more general case, Rubin and Gómez-Hernández (1990) obtained similar results derived using a perturbation approach valid only when both $\gamma(D, D)$ and $\sigma^2_{\ln k}\gamma(D, D)$ are small. For values of $\gamma(D, D)$ and $\sigma^2_{\ln k}$ typical of the present study, Rubin and Gómez-Hernández's results are considerably in error.

The function $\gamma(\cdot, \cdot)$ is referred to as the variance function and is shown in Appendix I for the random field considered in this paper, among others. It is an alternative way of characterizing the second-order information of a random field and can be obtained directly by integrating the covariance function (Vanmarcke 1984). In that most engineering properties are obtained as local averages of a spatially random field (i.e., permeability is measured using a finite volume sample, concrete strength is measured using a finite volume cylinder, etc.), the variance function is pertinent as it defines how the variability of a sample reduces as the averaging domain increases, thus demonstrating scale effects succinctly. The reduction in variance as averaging increases is a well known principle in statistics; intuitively, though, it may be understood more completely by considering the simple example of a row-boat near a supertanker on the surface of the (rough) ocean. Because the supertanker averages a much larger area of the ocean's surface, the variability in its motion is typically far less than that of the row-boat. They have the same mean displacement, but the supertanker has much smaller displacement variance (dynamic effects serve to enforce the averaging).

The variance function equals 1 when no averaging is performed and reduces monotonically to 0 as the averaging region increases. Specifically, if X_D is the random variable obtained by averaging the stationary random field X, with variance σ^2_X, over the domain $D \times D$ (in two-dimensions), then the variance of X_D is given by

$$\sigma^2_{X_D} = \sigma^2_X\gamma(D, D) \tag{11}$$

The parameter D = size of averaging region. In the case of (9), D refers to the side length of a two-dimensional square flow regime studied in Fenton and Griffiths (1993); thus the flow is affected by the average permeability in a domain of size $D \times D$. The mean and variance of flow through such a two-dimensional domain is expected to depend in some way on the reduction in variance due to averaging over the domain, leading to the results given by (9) and (10).

The shapes of the functions given by (9) are similar to those seen in Fig. 4, suggesting that these functions can be used to predict the mean and variance of log-flow through an earth dam if an effective value of D can be found to characterize the flow through the dam. Thus the task is to find the dimension D_{eff} of an equivalent two-dimensional square domain whose log-flow rate statistics (at least the mean and variance) are approximately the same as observed in the earth dam. One possible estimate of this effective dimension is

$$D_{\text{eff}} = \sqrt{\frac{A_{\text{wet}}}{\overline{Q}}} \tag{12}$$

where A_{wet} = earth dam area (in-plane) under free surface through which flow takes place, i.e., excluding unsaturated soil above free surface; and $\overline{Q}$ = nondimensionalized flow rate through the earth dam obtained with $K(\underline{x}) = \mu_k$ everywhere, that is

$$\overline{Q} = \frac{Q_{\mu_k}}{\mu_k H_{\text{eff}} z} \tag{13}$$

where H_{eff} = effective fluid head; and z = out-of-plane thickness of dam, which, for a two-dimensional analysis, is 1.0. Although it would appear reasonable to take H_{eff} as the average hydraulic head over the upstream face of the dam, it turns out to be better to take $H_{\text{eff}} = y_h/3$, the elevation of the centroid of the pressure distribution, where y_h = upstream water head

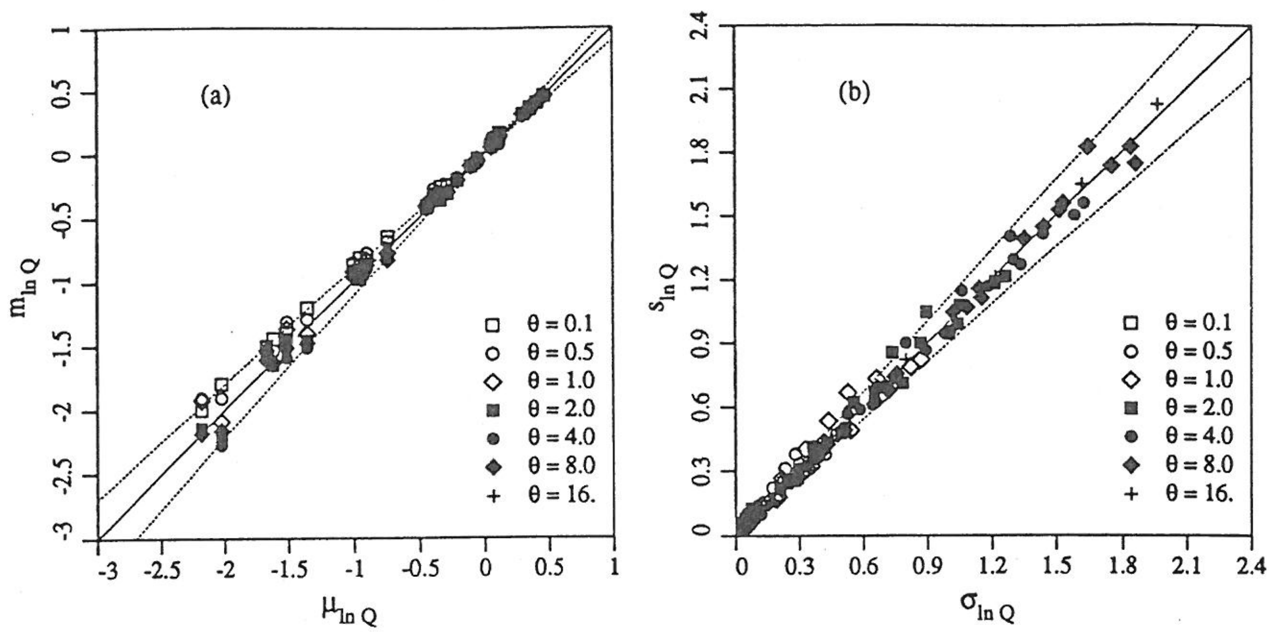

FIG. 7. Comparison of Following: (a) Mean; (b) Standard Deviation Statistics Derived via Simulation and as Predicted by (9)

(and overall height of dam). Substitution of (13) into (12), along with this choice of H_{eff} gives

$$D_{eff} = \sqrt{\frac{A_{wet}\mu_k y_h z}{3Q_{\mu_k}}} \quad (14)$$

This equation can then be used in (10) to estimate the desired flow rate statistics.

Fig. 7 illustrates the agreement between the mean and standard deviations derived via simulation and predicted using (14) in (9) for all four earth dam geometries shown in Fig. 1. The dashed lines in Fig. 7 denote the ±10% relative error bounds. It can be seen that most of the predicted statistics match those obtained from simulation quite well, in terms of absolute errors. A study of relative errors shows that 90% of the cases studied had relative errors less than 20% for both the prediction of the mean and standard deviation. There is no particular bias in the errors with respect to over- versus underestimation.

Admittedly, the effective dimension approach cannot properly reflect the correlation structure of the actual dam through a square domain approximation; if the dam width is significantly greater than the dam height (as in Dam 4), then the correlation between permeabilities at the top and bottom will generally be higher than from left edge to right edge. An equivalent square domain will not capture this. Thus, the effective dimension approach adopted here is expected to perform less well for long narrow flow regimes combined with scales of fluctuation approaching and exceeding the size of the dam. In fact, for the prediction of the mean, the simulation results belie this statement in that Dam 4 performed much better than Dams 1, 2, or 3. For the prediction of the standard deviation, Dam 4 performed the least well, perhaps as expected. Nevertheless, overall the results are encouraging.

Thus, the effective dimension approach can be seen to give reasonable estimates of the mean and variance of log-flow rates through the dam in most cases. To compute these estimates, the following steps must be performed:

1. Perform a single finite element analysis using $K(\underline{x}) = \mu_k$ throughout the earth dam to determine Q_{μ_k} and the area of the dam below the free surface, A_{wet}
2. Estimate the effective dam dimension using (14)
3. Compute the local average variance reduction factor, $\gamma(D_{eff}, D_{eff})$, corresponding to the random field used to model the log-permeability field (see Appendix I)
4. Estimate the mean and variance of log-flow through the dam using (9) (these values can be used directly in the lognormal distribution to compute probability estimates)

CONCLUSIONS

Although only a limited set of earth dam geometries are considered in the present paper, it should be noted that the stochastic response of a dam is dependent only on the ratio of the scale of fluctuation to the dam dimensions, for given dam shape and type of random field. For example, consider two earth dams with the same overall shapes and permeability statistics, μ_k and σ_k^2. If the second of the two dams is of twice the size and has twice the scale of fluctuation as the first, then the second will have twice the flow rate mean and standard deviation as the first and they will have identical normalized drawdown statistics. Similarly, the results shown here are easily scaled for different values of μ_k; the important parameter as far as the stochastic response is concerned is the coefficient of variation, σ_k/μ_k [or equivalently $\sigma_{\ln k}^2 = \ln(1 + \sigma_k^2/\mu_k^2)$]. These properties can be used to confidently employ the results of this paper on earth dams of arbitrary dimension and mean permeability.

For scales of fluctuation that are small relative to the size of the dam, the simulation results indicate the following:

1. The flow through the dam is well represented using only the estimated mean flow rate m_Q, that is, the flow rate variance is small
2. The mean flow rate falls rapidly as $\sigma_{\ln k}^2$ increases
3. The free surface profile will be relatively static and can be estimated confidently from a deterministic analysis [the simulation results imply that for both small and large scales of fluctuation (relative to the dam size), the drawdown variability is small and the Monte Carlo analysis could proceed using a fixed free surface found from the deterministic analysis, avoiding the need to iterate on each realization]

As the scale of fluctuation becomes larger, the mean flow rate does not fall as rapidly with increasing $\sigma_{\ln k}^2$ while the variability of the flow rate from one realization to the next increases significantly. The variability in the free surface location reaches a maximum for intermediate scales of fluctuation, apparently for scales of the order of the earth dam size.

The computation of estimates of the mean and variance of flow rates through an earth dam using (9) allow designers and planners to avoid full scale Monte Carlo simulations and can be used to approximately address issues regarding earth dam flow rate probabilities via a lognormal distribution. If more accurate estimates of these quantities are desired, particularly for scales of fluctuation approaching or greater than the dam size, then a full scale Monte Carlo simulation is currently the only viable choice. In that the mean, variance, and scale of fluctuation parameters of the permeability field, as estimated from the field, are themselves quite uncertain, the approximate estimate of the flow rate statistics may be quite appropriate in any case.

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APPENDIX I. VARIANCE FUNCTIONS

The variance function corresponding to the correlation function (3) cannot be determined analytically but is approximated by Vanmarcke (1984) to a reasonable degree of accuracy as

$$\gamma(\Delta_1, \Delta_2) = \frac{1}{2}\left[\gamma(\Delta_1)\gamma(\Delta_2|\Delta_1) + \gamma(\Delta_2)\gamma(\Delta_1|\Delta_2)\right] \quad (15a)$$

where

$$\gamma(\Delta_i) = \left[1 + \left(\frac{\Delta_i}{\theta_i}\right)^{3/2}\right]^{-2/3}; \quad \gamma(\Delta_i|\Delta_j) = \left[1 + \left(\frac{\Delta_i}{\theta_j^i}\right)^{3/2}\right]^{-2/3} \quad (15b,c)$$

$$\theta_j^i = \theta_i\left\{\frac{\pi}{2} + \left(1 - \frac{\pi}{2}\right)\exp\left[-\left(\frac{2\Delta_j}{\pi\theta_j}\right)^2\right]\right\} \quad (15d)$$

and where θ_i = directional scale of fluctuation (in the present study, $\theta_1 = \theta_2 = \theta_k$).

The variance function corresponding to the separable Gaussian correlation function

$$\rho(\underline{\tau}) = \exp\left[-\frac{\pi}{\theta^2}(\tau_1^2 + \tau_2^2)\right] \quad (16)$$

is given by the product

$$\gamma(\Delta_1, \Delta_2) = \gamma(\Delta_1)\gamma(\Delta_2) \quad (17)$$

of the corresponding 1D variance functions

$$\gamma(\Delta_i) = \frac{\theta_i^2}{\pi\Delta_i^2}\left[\frac{\pi|\Delta_i|}{\theta_i}\operatorname{erf}\left(\frac{\sqrt{\pi}|\Delta_i|}{\theta_i}\right) + \exp\left(-\frac{\pi\Delta_i^2}{\theta_i^2}\right) - 1\right] \quad (18)$$

Note that this result contained typographical errors in the paper by Fenton and Griffiths (1993).

Finally, the variance function corresponding to the correlation function (also separable)

$$\rho(\underline{\tau}) = \exp\left[-\frac{2}{\theta}(|\tau_1| + |\tau_2|)\right] \quad (19)$$

is the product

$$\gamma(\Delta_1, \Delta_2) = \gamma(\Delta_1)\gamma(\Delta_2) \quad (20)$$

of the corresponding 1D variance functions

$$\gamma(\Delta_i) = \frac{\theta_i^2}{2\Delta_i^2}\left[\frac{2|\Delta_i|}{\theta_i} + \exp\left(-\frac{2|\Delta_i|}{\theta_i}\right) - 1\right] \quad (21)$$

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APPENDIX III. NOTATION

The following symbols are used in this paper:

- A_{wet} = area of flow regime under free surface;
- D = side dimension of square flow domain;
- D_{eff} = effective earth dam dimension;
- $\operatorname{erf}\{\cdot\}$ = error function;
- $G(\underline{x})$ = mean zero, unit variance, Gaussian log-permeability field;
- H_{eff} = effective fluid head acting on dam;
- $K(\underline{x})$ = permeability field;
- $m_{\ln Q}$ = mean of log-flow rate estimated via simulation;
- m_Q = flow rate mean estimated via simulation;
- m_Y = normalized drawdown mean estimated via simulation;
- Q = total flow rate;
- $\bar{Q}$ = nondimensionalized total flow rate;

Q_{μ_k} = flow rate obtained with $K(\underset{\sim}{x}) = \mu_k$ everywhere;
$\underset{\sim}{q}$ = specific discharge vector (at a point);
$s_{\ln Q}$ = standard deviation of log-flow rate estimated via simulation;
s_Q = flow rate standard deviation estimated via simulation;
s_Y = normalized drawdown standard deviation estimated via simulation;
$\underset{\sim}{x}$ = spatial coordinate;
$\tilde{Y}$ = normalized drawdown (elevation of downstream flow exit point);
y_h = height of earth dam;
z = dam thickness in out-of-plane direction;
α = parameter of Beta distribution;
β = parameter of Beta distribution;
ϕ = hydraulic head;
$\gamma(\cdot, \cdot)$ = variance function (variance reduction due to local averaging);
μ_k = permeability mean;
$\mu_{\ln k}$ = mean of log-permeability;
$\mu_{\ln Q}$ = predicted mean of log-flow rate;
μ_Q = predicted flow rate mean;
θ, θ_k = scale of fluctuation of permeability fields;
$\rho(\cdot)$ = correlation function;
$\sigma^2_{\ln k}$ = variance of log-permeability;
$\sigma^2_{\ln Q}$ = predicted variance of log-flow rate;
σ^2_Q = predicted flow rate variance;
σ^2_k = permeability variance; and
$\underset{\sim}{\tau}$ = spatial separation or lag vector.