Probabilistic Settlement Analysis by Stochastic and Random Finite-Element Methods

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Abstract: The paper discusses finite element models for predicting the elastic settlement of a strip footing on a variable soil. The paper then goes on to compare results obtained in a probabilistic settlement analysis using a stochastic finite element method based on first order second moment approximations, with the random finite element method based on generation of random fields combined with Monte Carlo simulations. The paper highlights the deficiencies of probabilistic methods that are unable to properly account for spatial correlation.

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Introduction

A key objective of a probabilistic geotechnical analysis is to answer the following question: given statistical data (means, standard deviations, spatial correlation lengths, and cross correlations) relating to geotechnical input parameters (e.g. strength parameters c', $\tan\phi'$, seepage parameters k, and settlement parameters E) what is the probability of (usually undesirable) events occurring (e.g. bearing failure, excessive seepage, and excessive settlement)?

The finite element method offers great potential for tackling the probabilistic question posed in the first paragraph and has tended to be used in one of two modes: firstly under the general heading of the stochastic finite element methods (SFEMs) (e.g., Baecher and Ingra 1981; Vanmarcke and Grigoriu 1983; Ghanem and Spanos 1991; Haldar and Mahadevan 2000; and Sudret and Der Kiureghian 2002), where the finite element method has been combined with a truncated Taylor series, and secondly the random finite element method (RFEM), as developed by the authors (e.g., Griffiths and Fenton 1993; Fenton and Griffiths 1993), in which random fields are combined with the finite element method through Monte Carlo simulations.

Both methods lead to estimates of the mean and standard deviation of output events although the more general Monte Carlo approach has the advantage of generating the entire distribution of output events or of estimating probabilities directly by counting the number of times the event occurs as a proportion of the total number of simulations. In order to make probabilistic statements from the SFEM approach, the mean, and standard deviation must be combined with the assumption of a probability density function (e.g. Gaussian).

The Taylor series approach frequently involves truncation after the first order terms, leading to the well established first order second moment (FOSM) method. The method can estimate means and variances (or second moments) of functions of random variables, and has been described by numerous investigators in relation to geotechnical analysis (e.g., Harr 1987; Duncan 2000; and Baecher and Christian 2003).

In this paper we take a problem of elastic foundation settlement, and compare the approximate FOSM method as implemented in SFEM, with a general approach to the same problem using RFEM. Issues of relative accuracy and efficiency will be discussed.

Problem Statement

Assuming consistent units throughout, the problem considered in this paper as shown in Fig. 1 is of a smooth rigid strip footing of width $B=1.0$ subjected to a centrally applied vertical force $P=1.0$. The soil domain has a depth $H=2.0$ and the boundary conditions assume plane strain, with perfectly rough conditions at the bottom of the layer. The stiffness of the soil layer is defined by Young's modulus E, which in the probabilistic analyses will be assumed to be a lognormally distributed random field with mean μ_E, standard deviation σ_E, and (isotropic) spatial correlation length $\theta_{\ln E}$. The spatial correlation length $\theta_{\ln E}$ is the distance over which values of the log of Young's modulus can be expected to have significant positive correlation and may be conveniently nondimensionalized with respect to the footing width as $\Theta_E=\theta_{\ln E}/B$. Since Young's modulus is lognormal, the spatial correlation length is defined with respect to the underlying normal distribution of $\ln E$. The definition of correlation length will be discussed in more detail later in the paper. Poisson's ratio is constant throughout at $\upsilon=0.25$. The key output parameter in this study is the vertical footing settlement δ. Following some initial deterministic validations, the paper will focus on estimating the mean μ_δ and standard deviation σ_δ of this settlement by different finite element-based methods.

Deterministic Analysis

Some initial deterministic analyses were performed by finite elements to demonstrate the influence of boundary conditions and

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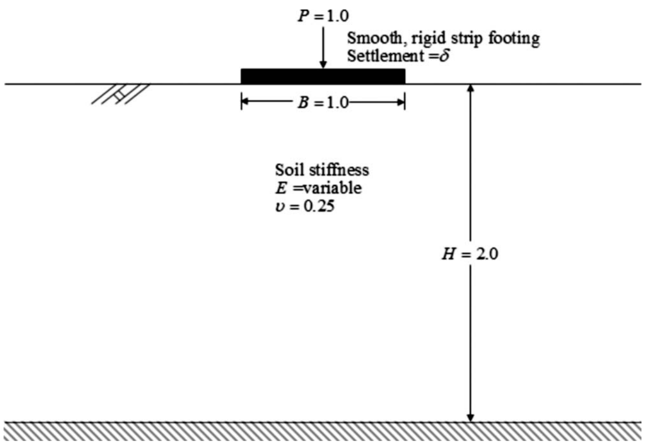

Fig. 1. Footing problem to be considered in this paper

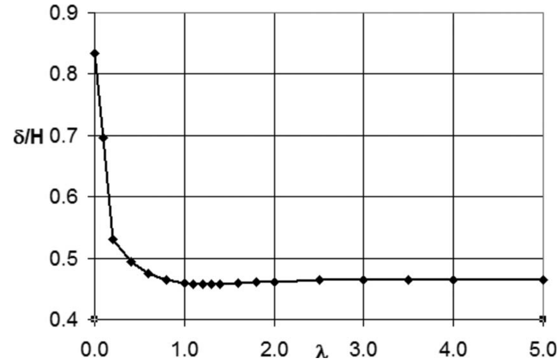

Fig. 3. Influence of side distance λB on settlement ratio δ/H

Young's modulus variability on footing settlement. In order to make direct comparisons between SFEM and RFEM later in the paper, a mesh of square four-node elements was used, as shown in Fig. 2. All runs used the public domain software of Smith and Griffiths (2004), with the smooth rigid footing modeled by "tying" the vertical freedoms at the nodes beneath the footing location.

Boundary Distance

Several runs were performed with uniform properties to assess the influence of side distance as a function of footing widths via the parameter λ, as shown in Fig. 2. Fig. 3 gives a plot of λ versus δ/H where δ/H is the settlement δ of the footing, nondimensionalized with respect to the depth of the soil layer H. The results indicate a steep reduction in settlement in the range $0<\lambda<1.0$, and reasonably constant settlement values for $\lambda>1.0$. The special case of $\lambda=0$ corresponds to confined one-dimensional compression which leads to a settlement given exactly by

$$\delta/H = \sigma_v(1+\upsilon)(1-2\upsilon)/E/(1-\upsilon) = 0.833 \qquad (1)$$

where $\sigma_v=P/B=1.0$=applied vertical stress. The probabilistic studies described later in this paper used $\lambda=1.6$, which corresponds to the plane strain mesh shown in Fig. 2, which has 840 square four-node elements and 1,786 degrees of freedom. With this value of λ, the boundaries will have little effect on predicted settlement.

Influence of Variable Stiffness

A brief study was performed to qualitatively assess the influence of variable soil stiffness on footing settlement. Fig. 4 shows a mesh in which Young's modulus takes two values, $E_{\max}$ and $E_{\min}$, that alternate from one element to the next in a checkerboard fashion, while maintaining the mean value at

$$\mu_E = (E_{\max}+E_{\min})/2 = 1.0 \qquad (2)$$

Poisson's ratio was fixed at $\upsilon=0.25$ and the rigid footing was subjected to a unit load as in the previous example. It may be noted that in this demonstration, a footing of width $B=1.1$ was assumed with an odd number of elements to guarantee symmetry.

The results shown in Fig. 5 for six different ($E_{\max}$, $E_{\min}$) combinations indicate that the less-stiff elements have a greater influence than the stiffer elements on the overall response. The ordinate is plotted in terms of the displacement magnification factor compared with what would have been obtained with a uniform soil with $E_{\max}=E_{\min}=1.0$ throughout.

The greatest magnification obtained was about 1.05 corresponding to the case where $E_{\max}=2.0$ and $E_{\min}=0.0$. In this extreme case, the white squares in Fig. 4 would correspond to voids with zero stiffness.

Probabilistic Analysis

The results of the previous section have indicated that variable soil stiffness leads to greater settlement than would be obtained in a uniform soil with the mean stiffness throughout. The remainder of this paper will introduce soil variability into the footing settle-

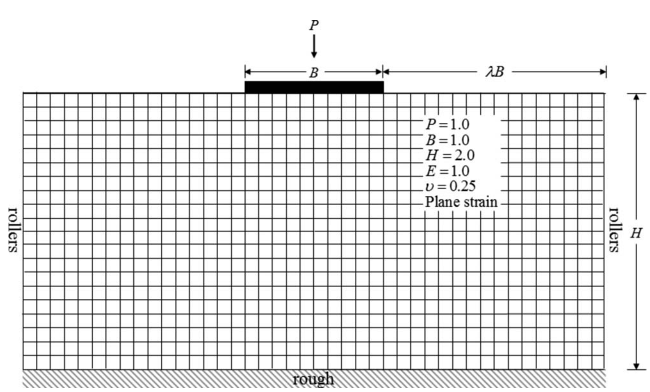

Fig. 2. Typical mesh of four-node square elements (840 elements, 1786 degrees of freedom)

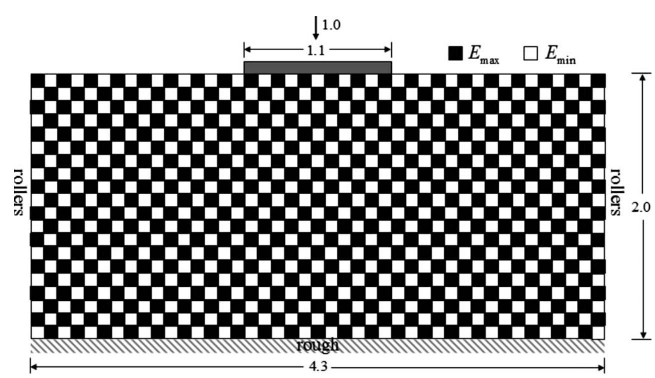

Fig. 4. Checkerboard study involving alternating Young's modulus values

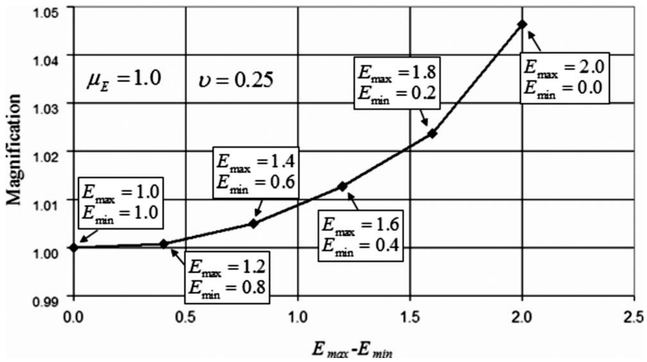

Fig. 5. Influence of alternating Young's modulus variability in checkerboard study

ment problem in a more systematic way by describing the soil stiffness statistically. In this approach the value of Young's modulus assigned to each element in the finite element mesh is treated as a random variable. We will start with the SFEM and a brief review of the underlying theory involving multiple random variables, i.e., random fields.

Review of the FOSM Method for Multiple Random Variables

Consider a function $f(X_1, X_2, \ldots, X_n)$ of n correlated random variables with means μ_{X_i}, $i=1,2,\ldots,n$ and variances $\sigma^2_{X_i}$, $i=1,2,\ldots,n$. To a first order of accuracy the mean of the function is given by

$$\mu_f = E[f(X_1,X_2,\cdots,X_n)] \approx f(\mu_{X_1},\mu_{X_2},\ldots,\mu_{X_n}) \tag{3}$$

and its variance by

$$\sigma^2_f = \mathrm{Var}[f(X_1,X_2,\cdots,X_n)] \approx \sum_{i=1}^{n}\sum_{j=1}^{n}(\partial f/\partial x_i)(\partial f/\partial x_j)\mathrm{Cov}[X_i,X_j] \tag{4}$$

$\mathrm{Cov}[X_i,X_j]$ is the covariance between X_i and X_j defined as

$$\mathrm{Cov}[X_i,X_j] = \rho_{ij}\,\sigma_{X_i}\sigma_{X_j} \tag{5}$$

where ρ_{ij} is the correlation coefficient between X_i and X_j.

The first derivatives in Eq. (4) are computed at the mean values $(\mu_{X_1},\mu_{X_2},\ldots,\mu_{X_n})$ and can be evaluated numerically or, if a functional form exists, analytically.

SFEM Using FOSM

We now reconsider the settlement problem as shown in Fig. 6 with a random Young's modulus. Let Young's modulus be a lognormally distributed random field with mean and standard deviation given by μ_E and σ_E, respectively, and spatial correlation length $\theta_{\ln E}$. In this paper we will assume a Markov correlation function given by

$$\rho_{\ln ij} = \exp(-2\tau_{ij}/\theta_{\ln E}) \tag{6}$$

where τ_{ij} denote (absolute) distance between two points i and j in the random field. For example, from Eq. (6), the log of Young's modulus at two points spaced $\theta_{\ln E}$ apart would have a correlation coefficient of $\rho_{\ln ij} \approx 0.14$. Points closer together would have higher values of $\rho_{\ln ij}$ and vice versa.

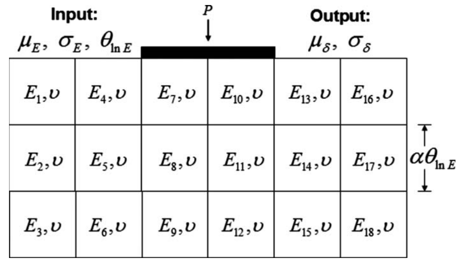

Fig. 6. Schematic of mesh with random Young's modulus assigned to each element

The random variables in this case are the values of Young's modulus assigned to each element in the mesh. For illustrative purposes Fig. 6 shows a rather coarse mesh, but the case studies described later will use the mesh shown in Fig. 2.

Based on a conventional elastic finite element analysis with n elements, the computed settlement of the rigid footing depends on the Young's modulus of each element, thus

$$\delta = f(E_1,E_2,\ldots,E_n) \tag{7}$$

hence from Eqs. (3) and (4), the mean settlement is given by

$$\mu_\delta = E[f(E_1,E_2,\ldots,E_n)] \approx f(\mu_{E_1},\mu_{E_2},\ldots,\mu_{E_n}) \tag{8}$$

and the variance by (see e.g., Fenton and Griffiths 2008)

$$\sigma^2_\delta = \mathrm{Var}[f(E_1,E_2,\ldots,E_n)] \approx \sum_{i=1}^{n}\sum_{j=1}^{n}(\partial\delta/\partial E_i)(\partial\delta/\partial E_j)\mathrm{Cov}[E_i,E_j] \tag{9}$$

In order to be consistent with the RFEM to come later, the input "point" statistics (μ_E,σ_E) have been adjusted inside the SFEM program to account for local averaging over each element. Thus the actual statistics used in the SFEM analysis are the locally averaged ones given by (μ_{E_A},σ_{E_A}). Local averaging deliver statistically consistent properties, but has nothing to do the "discretization error" inherent in any finite element analysis. If we define the side length of each square finite element in Fig. 6 as $\alpha\theta_{\ln E}$, two-dimensional local averaging, which is applied to the underlying normal distribution of $\ln E$, is expressed through a variance reduction factor $0<\gamma<1$ defined

$$\gamma = \sigma^2_{(\ln E)_A}/\sigma^2_{\ln E} \tag{10}$$

where $\sigma^2_{\ln E_A}$=locally averaged value of the variance of $\ln E$, accounting for spatial correlation length and element size. The variance reduction factor can be obtained from the expression

$$\gamma = (4/\alpha^4)\int_0^\alpha\int_0^\alpha(\alpha-x)(\alpha-y)e^{-2\sqrt{x^2+y^2}}dxdy \tag{11}$$

which can be evaluated by numerical integration.

Local averaging of a lognormal distribution of E causes both the mean and standard deviation to be reduced according to the following algorithm:

1. Given a lognormally distributed random variable E with point statistics μ_E and σ_E $(v_E=\sigma_E/\mu_E)$, and spatial correlation length $\theta_{\ln E}$;
2. Compute the underlying normal distribution parameters of $\ln E$ from $\mu_{\ln E}=\ln\mu_E-0.5\ln(1+v_E^2)$ and $\sigma_{\ln E}=\sqrt{\ln(1+v_E^2)}$;
3. Obtain α, and hence compute the variance reduction factor γ from Eq. (11);

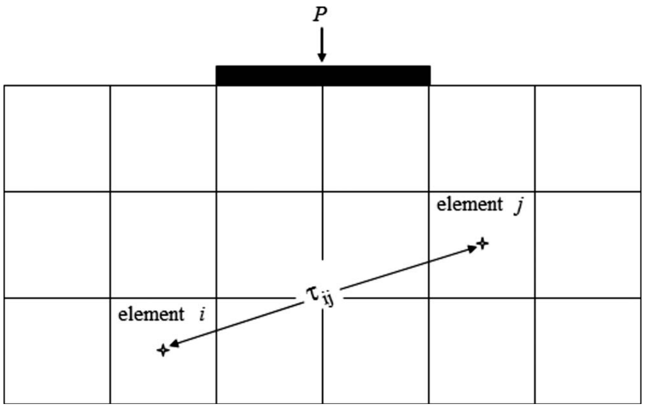

Fig. 7. Centroidal distance used in calculation of covariance matrix

4. Compute the locally averaged standard deviation from $\sigma_{(\ln E)_A} = \sigma_{\ln E}\sqrt{\gamma}$; and
5. Retrieve the locally averaged lognormal parameters from $\mu_{E_A} = \exp\,(\mu_{\ln E} + 0.5\sigma^2_{(\ln E)_A})$ and $\sigma_{E_A} = \mu_{E_A}\sqrt{\exp(\sigma^2_{(\ln E)_A}) - 1}$.

Essentially, local averaging becomes more significant when α is large (mesh coarse relative to the spatial correlation length) and less significant when α is small (mesh refined relative to the spatial correlation length).

The mean settlement from Eq. (8) is easily obtained by a conventional elastic finite element analysis with all E values within the mesh set to the locally averaged mean. In order to compute the variance of settlement from Eqs. (9) the covariance and the derivatives must first be found.

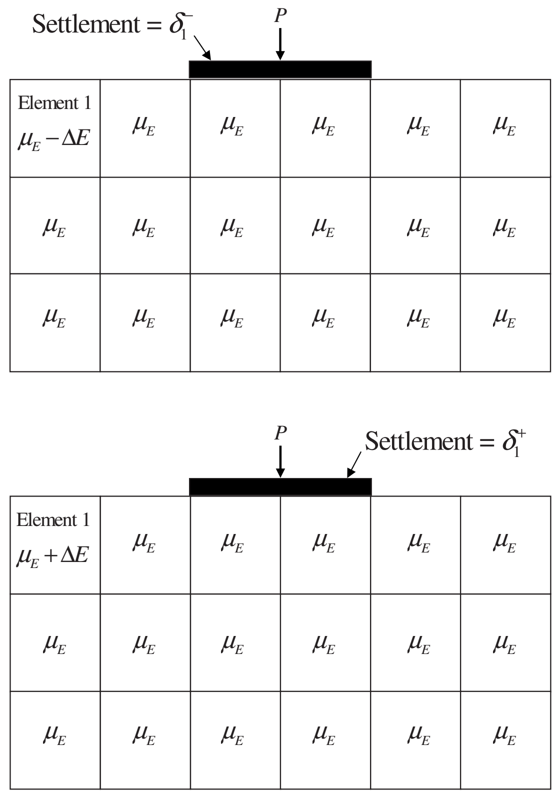

Fig. 8. Computation of settlement derivatives by numerical differentiation

Estimation of Covariance

In Eq. (6) we defined the correlation coefficient $\rho_{\ln ij}$ between the log of Young's modulus assigned to any two elements i and j, whose centroidal distance is given by τ_{ij}, as shown in Fig. 7. To estimate the covariance between actual Young's modulus values (not log), we need the correlation coefficient ρ_{ij} which can be obtained from the transformation (e.g., Vanmarcke 1983; Fenton and Griffiths 2008)

$$\rho_{ij} = [\exp(\rho_{\ln ij}\sigma^2_{\ln E}) - 1]/[\exp\,(\sigma^2_{\ln E}) - 1] \tag{12}$$

This leads to the "Covariance Matrix" needed by Eq. (9) as

$$C_{ij} = \mathrm{Cov}[E_i, E_j] = \rho_{ij}\sigma^2_E \tag{13}$$

Estimation of Derivatives

In general, derivative terms in Eq. (9) such as $\partial\delta/\partial E_i$ can be found using numerical differentiation. In this paper, where all the elements are square, an analytical approach is available because the element stiffness matrices can conveniently be expressed in closed form. Both methods will be briefly reviewed in the following sections.

Numerical Differentiation

In this case the derivatives are obtained by "perturbing" the stiffness of each element while anchoring the remaining elements at their mean value μ_E. As shown in Fig. 8, in order to estimate $\partial\delta/\partial E_1$ for Element 1, the footing settlement is computed twice; once corresponding to an increase in the mean stiffness $\mu_E + \Delta E$ to give δ_1^+, and once corresponding to a decrease in the mean stiffness $\mu_E - \Delta E$ to give δ_1^-. A central difference approximation then gives

$$\partial\delta/\partial E_1 \approx (\delta_1^+ - \delta_1^-)/(2\Delta E) \tag{14}$$

Following a similar numerical differentiation procedure for each of the elements, a mesh with n elements will therefore require a total of $2n+1$ independent elastic analyses in order to estimate the first order mean and standard deviation from Eqs. (8) and (9). The influence of the choice of increment ΔE is discussed later in this paper.

Analytical Differentiation

The numerical differentiation approach is quite general and can be used in any analysis, however if a functional form of the relationship between settlement δ and the Young's modulus of each element E_i is available, exact analytical differentiation can be performed. The global stiffness relationship for the mesh is given by

$$[\mathbf{K}]\{\boldsymbol{\delta}\} = \{\mathbf{F}\} \tag{15}$$

where $[\mathbf{K}]$ = global stiffness matrix; $\{\boldsymbol{\delta}\}$ = global displacement vector; and $\{\mathbf{F}\}$ = global force vector.

Differentiation of Eq. (15) with respect to each random variable E_i, $i = 1, 2, \ldots, n$ gives

$$\left(\frac{\partial}{\partial E_i}[\mathbf{K}]\right)\{\boldsymbol{\delta}\} + [\mathbf{K}]\frac{\partial}{\partial E_i}\{\boldsymbol{\delta}\} = \frac{\partial}{\partial E_i}\{\mathbf{F}\} \tag{16}$$

where $[\mathbf{K}]$ and $\{\boldsymbol{\delta}\}$=evaluated using μ_E in all elements. For constant loading

$$\frac{\partial}{\partial E_i}\{\mathbf{F}\} = \{\mathbf{0}\} \tag{17}$$

hence,

$$[\mathbf{K}]\left(\frac{\partial}{\partial E_i}\{\boldsymbol{\delta}\}\right) = -\left(\frac{\partial}{\partial E_i}[\mathbf{K}]\right)\{\boldsymbol{\delta}\} \tag{18}$$

and all the required derivatives can be obtained by solution of a set of equations (n times). The right-hand-side of Eq. (18) is obtained by assembling all the element derivative matrices, which for the square four-node elements with eight degrees of freedom as used in this study can be obtained in closed form as

$$\frac{\partial}{\partial E_i}[\mathbf{k}] = \frac{1}{24(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 4\upsilon_1 & -3 & -4\upsilon & 3\upsilon_2 & -2\upsilon_1 & 3 & -2\upsilon_3 & -3\upsilon_2 \\ & 4\upsilon_1 & -3\upsilon_2 & -2\upsilon_3 & 3 & -2\upsilon_1 & 3\upsilon_2 & -4\upsilon \\ & & 4\upsilon_1 & 3 & -2\upsilon_3 & 3\upsilon_2 & -2\upsilon_1 & -3 \\ & & & 4\upsilon_1 & -3\upsilon_2 & -4\upsilon & -3 & -2\upsilon_1 \\ & & & & 4\upsilon_1 & -3 & -4\upsilon & 3\upsilon_2 \\ & & & & & 4\upsilon_1 & -3\upsilon_2 & -2\upsilon_3 \\ & & & & & & 4\upsilon_1 & 3 \\ & & & & & & & 4\upsilon_1 \end{bmatrix} \tag{19}$$

where $\upsilon_1 = 3-4\upsilon$; $\upsilon_2 = 1-4\upsilon$; and $\upsilon_3 = 3-2\upsilon$.

The analyses described by Eqs. (15)–(18) involve the solution of 1,786 simultaneous equations $n+1$ times, where n is the number of elements in the mesh. This can be contrasted with the solution of the same number of simultaneous equations $2n+1$ times in the numerical version. Even more importantly, since the left-hand-side matrix $[\mathbf{K}]$ in the analytical version remains unchanged, it need only be factorized once, followed by $n+1$ forward and back-substitutions. The numerical version on the other hand cannot take advantage of factorization since each global stiffness matrix in Eq. (15) is different. In summary, the analytical version requires significantly less computational effort than the numerical version.

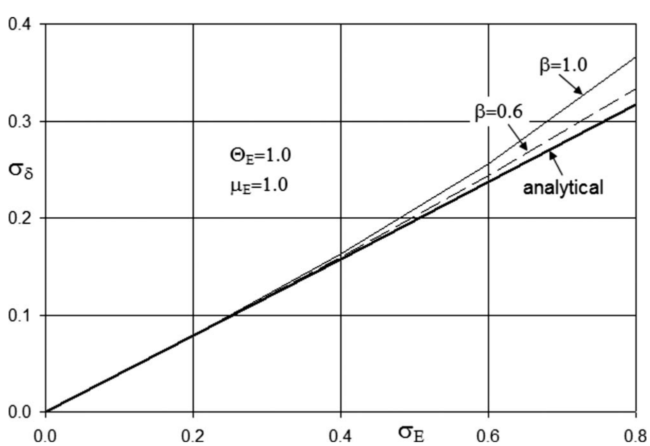

Fig. 9. Comparison of results obtained by numerical and analytical differentiations

Comparison of Results Using Numerical and Analytical Differentiation

Analytical differentiation is "exact" within the limitations of the finite element discretization, so it offers us the opportunity to assess the accuracy of the more general, but approximate numerical differentiation approach. The key parameter to be assessed here is the magnitude of the "perturbation" ΔE as used in Eq. (14). In order to compare numerical and analytical results, ΔE will be expressed as

$$\Delta E = m\sigma_E \tag{20}$$

where m represents the proportion of the standard deviation that the central difference formula deviates Young's modulus below and above the mean.

Fig. 9 shows the standard deviation of the output footing settlement as a function of the standard deviation of the input Young's modulus for two m values. In all cases the mean Young's modulus was fixed to $\mu_E=1.0$ and the dimensionless correlation length to $\Theta_E=1.0$.

The use of $m=1.0$ as advocated by some investigators [e.g., Duncan (2000)], implies $\Delta E=\sigma_E$ which by any standards represents a large perturbation. The $m=1.0$ result consistently gives a higher settlement variability than obtained by the analytical approach, although the difference is not that great considering it makes use of a central difference scheme that encompasses a range of two standard deviations. Results obtained with lower m values converged rapidly on the analytical solution. SFEM results presented in this paper used the analytical approach.

Review of the RFEM

The RFEM involves generating a random field of soil properties with controlled mean, standard deviation and spatial correlation length, which is then mapped onto a finite element mesh. A con-

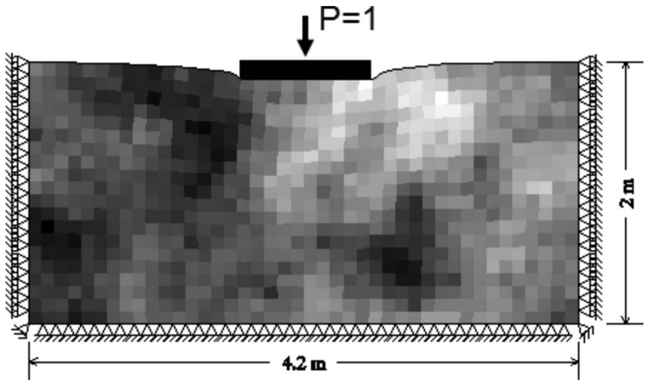

Fig. 10. Typical realization of an RFEM analysis showing the random field of Young's modulus (darker is stiffer)

ventional elastic finite element analysis using these properties is then performed to compute the footing settlement, after which the process is repeated many times using Monte Carlo simulations. In the Monte Carlo process, the underlying statistics of Young's modulus are held constant, however its spatial distribution and hence the computed settlement of the footing (under a constant load) is different at each simulation. Following a sufficient number of simulations, the output settlement becomes statistically stable and can be analyzed. A study involving RFEM analysis of single and multiple footings has previously been described by Paice et al. (1996) and Fenton and Griffiths (2002, 2005). In the current study 2,000 realizations were used for each parametric combination, leading in each case to an estimate of the mean and standard deviation of the footing settlement given by μ_δ and σ_δ, respectively. It should be noted that unlike the SFEM approach, this method gives a histogram of settlement values which can be fitted, if required, to an appropriate function (e.g., lognormal) for probabilistic interpretation. The program used in this study and many others for performing geotechnical analysis by RFEM is freely available from the website (http://www.mines.edu/~vgriffit/rfem).

The random field is generated using the local average subdivision (LAS) method (Fenton and Vanmarcke 1990) which takes full account of local averaging as previously described in Section 5. A convenient aspect of LAS is that the random field is generated over cells that are the same size as the finite elements, greatly facilitating the mapping of properties onto elements. A typical realization of an RFEM analysis is shown in Figure Fig. 10. The darker zones indicate stiffer soil, and the lighter zones indicate less-stiff soil.

Comparison and Discussion of SFEM and RFEM Results

The objective of this section is to compare the mean and standard deviation of footing settlement as estimated by the SFEM and RFEM methods (e.g., Herlyck 2005). In all cases the mean Young's modulus is held constant at $\mu_E=1$ while the standard deviation and spatial correlation length are varied in the range $0<\sigma_E<1$ and $0<\Theta_E<5$. In all cases, the mesh of Fig. 2 was used, and Poisson's ratio was held constant at $\upsilon=0.25$. For comparison purposes later, we will define the deterministic settlement $\delta_{det}=0.92$ as the footing settlement that would be obtained on a uniform soil with the stiffness set everywhere to the mean $\mu_E=1$.

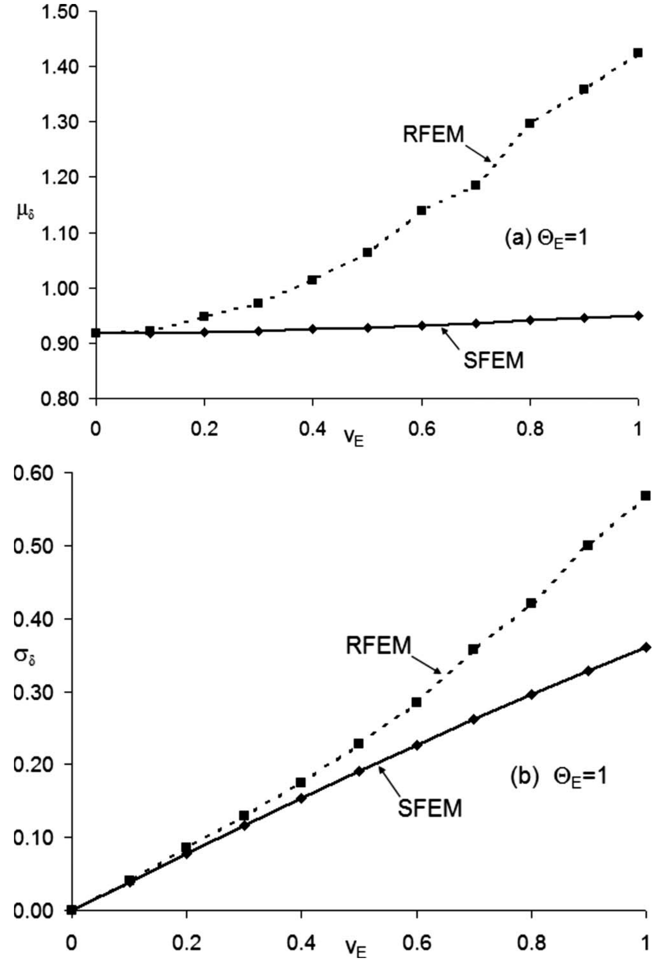

Fig. 11. Variation of (a) mean and (b) standard deviation of footing settlement versus coefficient of variation of input

In Fig. 11, the coefficient of variation of Young's modulus, $\upsilon_E=\sigma_E/\mu_E$ has been varied while fixing the spatial correlation length at $\Theta_E=1$. Both the mean (μ_δ) and standard deviation (σ_δ) of settlement as shown in Figs. 11(a and b), respectively, are underestimated by SFEM as compared with RFEM, with the difference growing consistently with the input coefficient of variation of Young's modulus (υ_E). These results indicate that SFEM will lead to unconservative (underestimates) of the probability of the settlement exceeding some allowable design threshold. The error in SFEM in both the mean and standard deviation plots is quite similar and reaches about 15% at $\upsilon_E=0.5$, which might be considered an upper-bound on stiffness variability for many soils (e.g., Lee et al. 1983). Matthies et al. (1997) observed a similar divergence between Monte Carlo and perturbation methods in a different application. The consistency of the underestimation in both settlement mean and standard deviation is further emphasized in Fig. 12, where the coefficient of variation ($\upsilon_\delta=\sigma_\delta/\mu_\delta$) of the predicted settlement by both methods is remarkably similar. In Fig. 13, the spatial correlation length of Young's modulus has been varied in the range $0<\Theta_E<5$ while fixing its coefficient of variation to $\upsilon_E=0.3$.

The influence of spatial correlation length on footing settlement in random soils using the RFEM has been discussed in detail in Fenton and Griffiths (2002, 2005). As $\Theta_E\rightarrow 0$, (1) the mean settlement tends to a value corresponding to a uniform soil with the stiffness set everywhere to the median Young's modulus

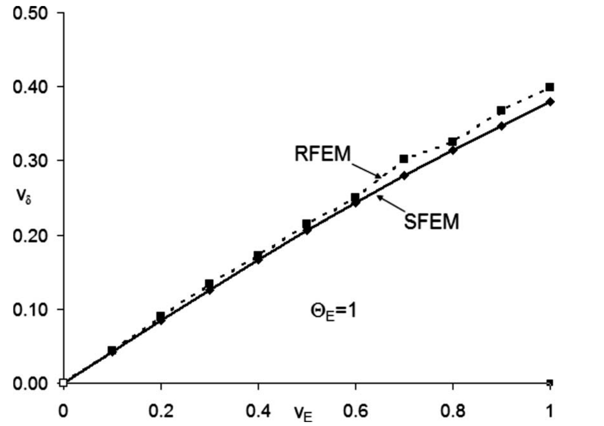

Fig. 12. Coefficient of variation of input (v_E) and output (v_δ)

given by $\exp(\mu_{\ln E})$. Since the median of a lognormal distribution is smaller than the mean by the factor $(1+v_E^2)^{1/2}$, the settlement in therefore bigger than δ_{det} by this same factor, and (2) the standard deviation tends to zero, thus

$$\mu_{\delta(\text{RFEM})} \rightarrow \delta_{\text{det}}(1+v_E^2)^{1/2}$$
$$\sigma_{\delta(\text{RFEM})} \rightarrow 0 \quad (21)$$

In this case $\mu_{\delta(\text{RFEM})} \rightarrow 0.96$ and $\sigma_{\delta(\text{RFEM})} \rightarrow 0$, which are the values given by the RFEM results in Figs. 13(a and b) as $\Theta_E \rightarrow 0$.

Both SFEM and RFEM are in agreement for $\Theta_E \rightarrow 0$, however for higher values of Θ_E, the methods give diverging results with SFEM consistently underestimating both the mean and standard deviation of settlement as shown in Figs. 13(a and b). Once more we can conclude that SFEM will lead to unconservative probabilistic estimates.

Closer inspection of Fig. 13(a) reveals that as $\Theta_E \rightarrow \infty$, the SFEM gives a falling mean settlement where $\mu_{\delta(\text{SFEM})} \rightarrow \delta_{\text{det}} = 0.92$, which is the deterministic result obtained in a uniform soil with Young's modulus set everywhere to the mean value of Young's modulus $\mu_E = 1.0$.

The RFEM results are more ragged due to the Monte Carlo simulations, however there is a clear trend as $\Theta_E \rightarrow \infty$ of increasing mean settlement. Fenton and Griffiths (2005) showed theoretically that due the reciprocal relationship between E and δ, as $\Theta_E \rightarrow \infty$

$$\mu_{\delta(\text{RFEM})} \rightarrow \delta_{\text{det}}(1+v_E^2)$$
$$\sigma_{\delta(\text{RFEM})} \rightarrow \mu_{\delta(\text{RFEM})} v_E \quad (22)$$

In this case $\mu_{\delta(\text{RFEM})} \rightarrow 1.00$ and $\sigma_{\delta(\text{RFEM})} \rightarrow 0.30$, which are the asymptotes to which the RFEM results in Figs. 13(a and b) are heading as $\Theta_E \rightarrow \infty$.

Fig. 12 demonstrated that SFEM and RFEM give the same coefficient of variation of the footing settlement, so the footing standard deviation asymptote as $\Theta_E \rightarrow \infty$ from SFEM is $\sigma_{\delta(\text{SFEM})} \rightarrow \mu_{\delta(\text{SFEM})} v_E = 0.28$, as shown in Fig. 13. The SFEM results underestimate the probability of design failure and are on the "unsafe" side. This conclusion would have been reached for any initial choice of Θ_E and v_E. The lower probability of failure is clearly due to the lower mean settlements consistently predicted by SFEM, but fundamentally, the shortcomings of SFEM as compared with RFEM, is that it is unable to directly model the influence of a spatially variable soil stiffness.

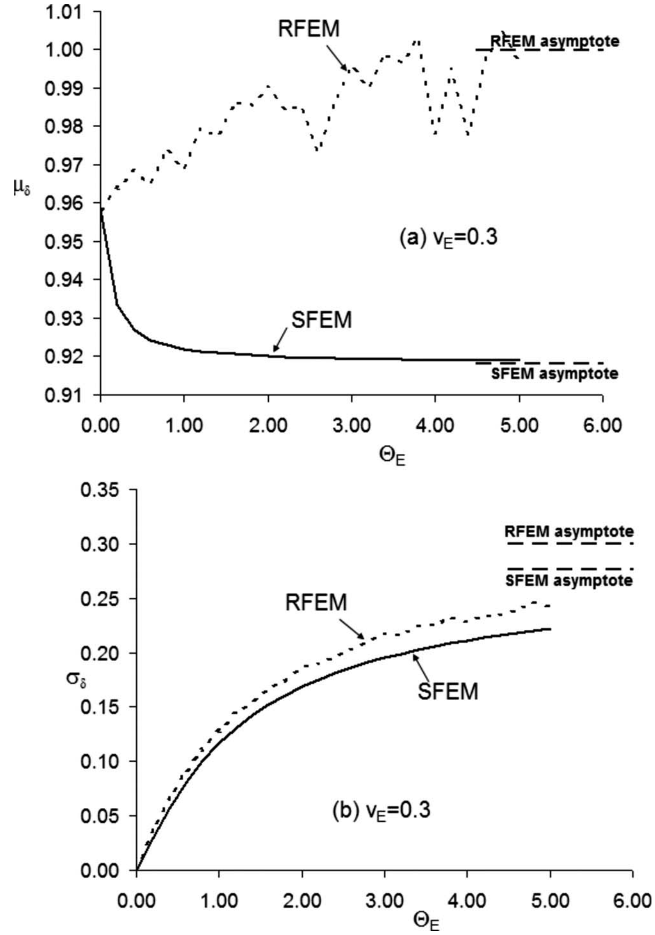

Fig. 13. Variation of (a) mean and (b) standard deviation of footing settlement versus spatial correlation length of Young's modulus

Concluding Remarks

The paper has presented results of finite element analyses of the settlement of a rigid footing on an elastic soil. First a sensitivity study demonstrated the influence of side boundary proximity and then the investigation was expanded to examine the role of variable stiffness on footing settlement. In a simple "checkerboard" study, it was shown that if stiff and less-stiff elements are distributed systematically beneath the footing, the less-stiff zones dominate the response and the overall settlement increases. The paper then compared two probabilistic finite element methodologies for the stochastic analysis of footing settlement. Results from a SFEM based on FOSM approximations were compared with results from a RFEM which involved modeling the soil as a random field with Monte Carlo simulations.

The SFEM cannot directly model the influence of spatial variability nor the nonsymmetric lognormal distribution. This shortcoming was particularly highlighted when v_E was held constant and the spatial correlation length Θ_E gradually increased. In this case the mean settlements by the two methods went in opposite directions, with SFEM and RFEM predicting decreasing and increasing mean settlement, respectively. The divergence between the two means, as shown in the simple checkerboard study, is due to the fact that settlement is dominated by the low stiffness values in the random fields. The SFEM also consistently underestimated the standard deviation of footing settlement. In any subsequent probabilistic study of allowable settlement, the SFEM would inevitably lead to unconservative (underestimates) of the probability of design failure.

In conclusion, care must be taken when using the less general SFEM approach in probabilistic foundation settlement analysis While SFEM may give reasonable predictions for low input property variance, it will always lead to underestimates of the probability of design failure compared with the RFEM.

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Notation

The following symbols are used in this paper:

B = footing width;
C_{ij} = covariance between points i and j in the random field;
c' = cohesion;
E = Young's modulus;
E_i = Young's modulus in the ith element;
$E_{\max}$ = maximum Young's modulus;
$E_{\min}$ = minimum Young's modulus;
$\{\mathbf{F}\}$ = global force vector;
H = depth of soil layer;
i = simple counter;
$[\mathbf{K}]$ = global stiffness matrix;
k = permeability;
$[\mathbf{k}]$ = element stiffness matrix;
m = perturbation parameter during numerical differentiation;
n = number of random variables;
P = load on footing;
v_E = coefficient of variation of Young's modulus $(=\sigma_E/\mu_E)$;
v_δ = coefficient of variation of Young's modulus $(=\sigma_\delta/\mu_\delta)$;
X_i = ith random variable;
α = element size parameter;
γ = variance reduction factor;
ΔE = change in the Young's modulus during numerical differentiation;
δ = footing settlement;
$\{\boldsymbol{\delta}\}$ = global displacement vector;
δ_i^+ = footing settlement due to an increment of ΔE in element i;
δ_i^- = footing settlement due to a decrement of ΔE in element i;
$\delta_{\det}$ = deterministic settlement with the mean Young's modulus;
Θ_E = dimensionless spatial correlation length $(=\theta_{\ln E}/B)$;
$\theta_{\ln E}$ = spatial correlation length of log of Young's modulus;
λ = side distance parameter;
μ_E = mean of Young's modulus;
μ_{E_A} = locally averaged mean of Young's modulus;
μ_{E_i} = mean of Young's modulus in the ith element;
μ_f = mean of a function of random variables;
$\mu_{\ln E}$ = mean of log of Young's modulus;
$\mu_{(\ln E)_A}$ = locally averaged mean of log of Young's modulus;
μ_{X_i} = mean of the ith random variable;
μ_δ = mean of footing settlement;
$\mu_{\delta(\mathrm{RFEM})}$ = mean of footing settlement by RFEM;
$\mu_{\delta(\mathrm{SFEM})}$ = mean of footing settlement by SFEM;
ρ_{ij} = correlation coefficient between X_i and X_j;
$\rho_{\ln ij}$ = correlation coefficient between $\ln X_i$ and $\ln X_j$;
σ_E = standard deviation of Young's modulus;
σ_{E_A} = locally averaged standard deviation of Young's modulus;
σ_f = standard deviation of a function of random variables;
$\sigma_{\ln E}$ = standard deviation of log of Young's modulus;
$\sigma_{(\ln E)_A}$ = locally averaged standard deviation of log of Young's modulus;
σ_v = vertical stress on footing;
σ_{X_i} = standard deviation of the ith random variable;
σ_δ = standard deviation of footing settlement;
$\sigma_{\delta(\mathrm{RFEM})}$ = standard deviation of footing settlement by RFEM;
$\sigma_{\delta(\mathrm{SFEM})}$ = standard deviation of footing settlement by SFEM;
τ_{ij} = distance between points i and j in the random field;
υ = Poisson's ratio;
$\upsilon_1, \upsilon_2, \upsilon_3$ = functions of Poisson's ratio; and
ϕ' = friction angle.

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