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SYSTEM RELIABILITY OF SLOPES BY RFEM

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ABSTRACT

In a probabilistic slope stability analysis, the failure probability associated with the most critical slip surface (the one with the minimum reliability index) is known to be smaller than that for the system that comprises all potential slip surfaces. The first order reliability method (FORM) targets the minimum reliability index related to the critical slip surface, and thus cannot be used to predict the system reliability of slopes, except when all possible slip surfaces are perfectly correlated. It is shown in this paper that the random finite element method (RFEM), which uses elastoplastic finite elements combined with random field theory in a Monte-Carlo framework can accurately predict the system probability of failure (p_f) of slopes.

Key words: finite element method, random field, spatial correlation, system probability of failure, two-layer slope (**IGC:** E6)

INTRODUCTION

The probability of failure of a system is computed as:

$$p_f = P(g(X) \le 0) = \int_{g(X)\le 0} f(X)\mathrm{d}X \tag{1}$$

where $f(X)$ is the joint probability density function (PDF) of the input variables X and g is the limit state function which defines safe or unsafe performance. Limit states could relate to strength failure, serviceability failure, or anything else that describes unsatisfactory performance. The limit state function is customarily defined as

$$\begin{aligned} g(X) \ge 0 &\longrightarrow \text{Safe} \\ g(X) < 0 &\longrightarrow \text{Failure} \end{aligned} \tag{2}$$

By assuming the limit state function follows a normal distribution, the generalized reliability index β_g, is commonly used as an alternative measure of safety. It is defined as

$$\beta_g = \Phi^{-1}(1-p_f) \tag{3}$$

where $\Phi(\cdot)$ represents the cumulative density function (CDF) of the standard normal distribution.

The direct integration of Eq. (1) is usually impossible since many geotechnical problems do not have exact analytical solutions to the deterministic problem (e.g., the slope stability problem). There are several probabilistic methods, for example, the first-order second-moment (FOSM) and the first order reliability method (FORM), which obtain an approximation to the minimum reliability index (β_{min}) first and then obtain the p_f by

$$p_f = 1 - \Phi(\beta_{min}) \tag{4}$$

The FOSM method (e.g., Hassan and Wolff, 1999; Bhattacharya et al., 2003) and the FORM (e.g., Low 1996; Low et al., 1998; Xu and Low, 2006) are widely used in reliability analyses of slope problems. In FOSM, the mean and variance of the limit state function are approximated by a first-order Taylor series expansion about the mean values of the input random parameters that are characterized by their first two moments. A serious problem with FOSM is that the reliability index it delivers depends on how the limit state function is formulated, thus two people solving the same problem could obtain quite different results. FORM, on the other hand, is not affected by the formulation of the limit state function and computes a reliability index as the shortest distance (in standard deviations) from the equivalent mean-value point to the limit state surface, from which the probability of failure can be obtained from Eq. (4). By definition, FORM targets the minimum reliability index related to the critical slip surface, which may lead to an unrealistic estimation of system slope reliability. This is because there are many potential slip surfaces, each of which has a finite probability of failure associated with it.

Probabilistic slope stability analysis must be treated as a system reliability problem. Upper and lower bounds of system reliability have been described by Cornell (1967) and Ditlevsen (1979). As pointed out by Cornell (1967), a

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system's reliability is that of all potential slip surfaces, and the failure probability of a system will be larger than that for any single slip surface. The difference depends on the correlation between the failure probabilities of the different slip surfaces, for which no general formulation is available. It is, therefore, somewhat surprising that procedures for system reliability studies have rarely been developed for slopes. The only exceptions are the papers by Oka and Wu (1990) and Chowdhury and Xu (1995) which presented system reliability analysis for a particular slope in which the factors of safety (*FS*) of several slip surfaces were poorly correlated. The importance of system reliability of slopes was also noted by Mostyn and Li (1993) and Shinoda (2007).

This paper will use RFEM and FORM to study a two-layer slope reliability problem. The RFEM, first used in a slope stability application by Griffiths and Fenton (2000, 2004), was further developed herein to introduce a second random field enabling the probabilistic analysis of a slope consisting of two soil layers with different random properties. RFEM uses elastoplastic finite elements combined with the random field theory in a Monte-Carlo framework. The probability of failure is directly obtained by dividing the number of simulations which failed by the total number of simulations. It will be shown that the RFEM offers the only general way of predicting the system reliability of slopes. Most FORM applications described in the literature do not consider spatial variability, but some investigators have combined the FORM with Limit Equilibrium Methods (LEM) and random field theory (e.g., Babu and Mukesh, 2004; Low et al., 2007). The inherent nature of LEM, however, is that it leads to a critical failure surface, which in 2-d analysis appears as a line which could be non-circular. The influence of the random field is only taken into account along the line and is, therefore, effectively one-dimensional. A further disadvantage of FORM is that with no explicit limit state function, FORM must rely on the development of a response surface method (RSM) (e.g., Melchers, 1999; Xu and Low, 2006) involving curve fitting a function involving hundreds if not thousands of random variables. Not only is this impractical, but the authors are unable to find any literature where this has been attempted. Most FORM applications do not consider spatial correlation and those that do consider it, do so in 1-d version along a failure surface. The focus of this paper is to compare the more accepted version of FORM with RFEM in their abilities to generate unbiased estimates of failure probability of two-layer slopes consisting of different random soils.

The paper first investigates the influence of foundation strength on the *FS* of two-layer slopes. The FORM is then used to study the p_f of slopes showing that p_f does not change if including a foundation does not change the *FS*. Finally, the RFEM is used to study the influence of a random foundation on p_f. The results show that the system probability of failure of the two-layer slope is higher than the probability of failure of the embankment only, even if the strength of the foundation is high enough that no changes are made to *FS*.

DETERMINISTIC ANALYSES

The method and program of Griffiths and Lane (1999) was used to analyze the slopes in this section.

Undrained ($\phi_u = 0$) Slopes

An $\alpha = 26.6°$ (2:1 slope) undrained ($\phi_u = 0$) slope is considered with the slope profile shown in Fig. 1(a). The slope has height $H = 10.0$ m, soil unit weight γ_{sat} (or γ) = 20.0 kN/m^3, shear strength $c_{u1} = 30.6$ kPa (expressed in a dimensionless form given by $C_{u1} = c_{u1}/(\gamma_{sat} H) = 0.153$). The *FS* of the slope was found to be 1.25. The deformed mesh at failure is shown in Fig. 1(b).

Another two-layer slope with a similar geometry but including a foundation with depth ratio $D = 2$ as shown in Fig. 2(a) is further considered. The foundation was assumed to be undrained soil of the same unit weight γ_{sat} (or g) = 20.0 kN/m^3 but with a different shear strength, given by $c_{u2} = 45.8$ kPa ($C_{u2} = 0.229$). The *FS* of the two-layer slope was found to be also 1.25. The deformed mesh at failure is shown in Fig. 2(b). As shown by Griffiths and Lane (1999) for this case, if $C_{u2}/C_{u1} \geq 1.5$, the foundation strength has no influence on the *FS*. This is confirmed by varying C_{u2}/C_{u1} in the range of {0.25, 0.5, . . . , 2.5} and fixing $C_{u1} = 0.153$. The results are shown in Fig. 3. A

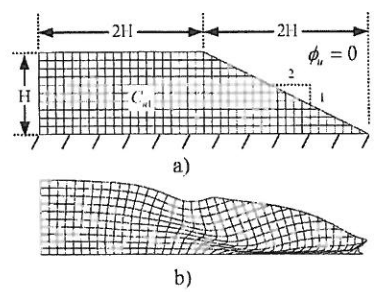

Fig. 1. Undrained slope without foundation

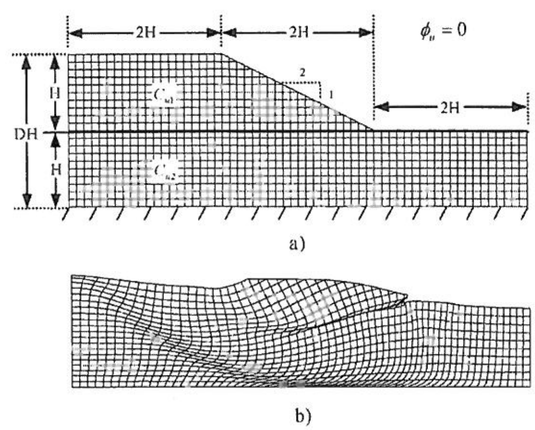

Fig. 2. Undrained two-layer slope

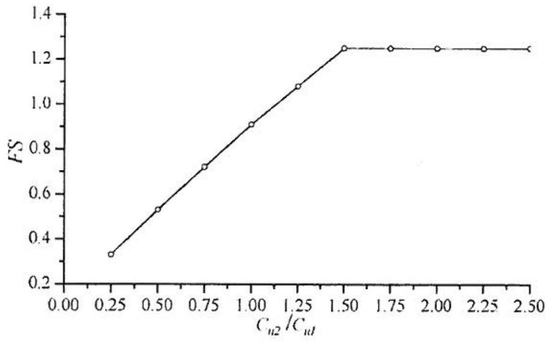

Fig. 3. Influence of C_{u2}/C_{u1} on the *FS* of undrained two-layer slopes ($C_{u1}=0.153$)

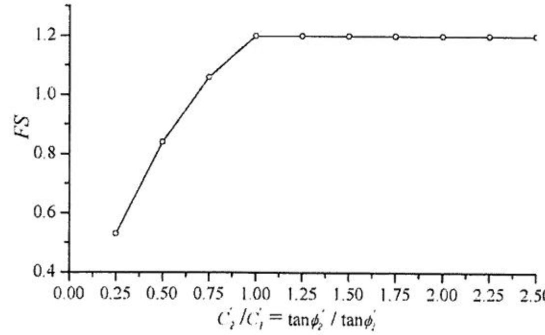

Fig. 6. Influence of $C'_2/C'_1=\tan\phi'_2/\tan\phi'_1$ on the *FS* of drained two-layer slopes ($C'_1=0.035$ and $\tan\phi'_1=0.364$)

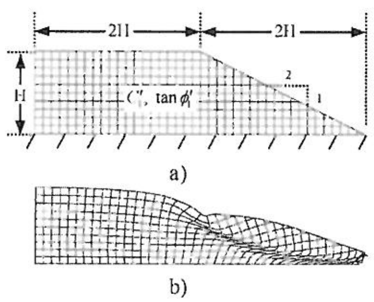

Fig. 4. Drained slope without foundation

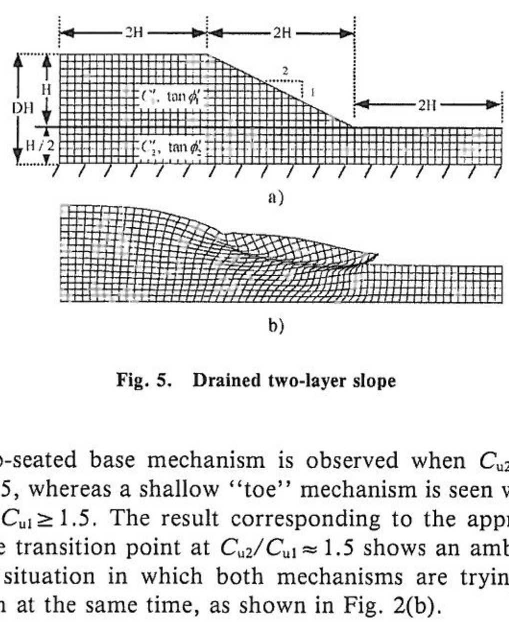

Fig. 5. Drained two-layer slope

deep-seated base mechanism is observed when $C_{u2}/C_{u1} \leq 1.5$, whereas a shallow "toe" mechanism is seen when $C_{u2}/C_{u1} \geq 1.5$. The result corresponding to the approximate transition point at $C_{u2}/C_{u1} \approx 1.5$ shows an ambiguous situation in which both mechanisms are trying to form at the same time, as shown in Fig. 2(b).

Drained Slopes

An $\alpha=26.6°$ (2:1 slope) drained slope is now considered with the slope profile shown in Fig. 4(a). The slope has height $H=10.0$ m, soil unit weight $\gamma=20.0$ kN/m^3, and shear strength parameter $c'_1=7.0$ kPa (expressed in the dimensionless form $C'_1=c'_1/(\gamma H)=0.035$) and $\tan\phi'_1=0.364$. The *FS* of the slope was found to be 1.20. The deformed mesh at failure is shown in Fig. 4(b).

Another two-layer slope with a similar geometry but with a foundation depth ratio of $D=1.5$, as shown in Fig. 5(a), is further considered. The foundation was assumed to have the same strength as the embankment. ($C'_1=C'_2=0.035$, $\tan\phi'_1=\tan\phi'_2=0.364$ and $\gamma=20.0$ kN/m^3) leading to $FS=1.20$. The deformed mesh at failure is shown in Fig. 5(b). The foundation makes no change to the slope *FS* if the shear strengths ratio $C'_2/C'_1=\tan\phi'_2/\tan\phi'_1$ is greater than one. This is confirmed by changing $C'_2/C'_1=\tan\phi'_2/\tan\phi'_1$ in the range of $\{0.25, 0.5, \ldots, 2.5\}$ and fixed $C'_1=0.035$ and $\tan\phi'_1=0.364$, as shown in Fig. 6.

PROBABILISTIC DESCRIPTIONS OF STRENGTH PARAMETERS

In this study, the shear strength parameters C_{u1}, C_{u2}, C'_1, C'_2, $\tan\phi'_1$ and $\tan\phi'_2$ are assumed to be random variables characterized statistically by lognormal distributions (i.e., the logarithms to the base *e* of the properties are normally distributed). The lognormal distribution will be applied at the point level. The lognormal distribution is one of many possible choices (e.g., Fenton and Griffiths, 2008), however, it offers the advantage of simplicity, in that it is arrived by a simple nonlinear transformation of the classical normal (Gaussian) distribution. Lognormal distributions guarantee that the random variable is always positive, and have been advocated and used by several other investigators in addition to the current authors, as a reasonable model for soil properties (e.g., Parkin et al., 1988; Parkin and Robinson, 1992; Nour et al., 2002; Massih et al., 2008). The RFEM methodology has been described in detail in other publications (e.g., Fenton and Griffiths, 2008), so only a brief description will be repeated here for the random variable C_{u1}. An identical procedure is applied to C_{u2}, C'_1, C'_2, $\tan\phi'_1$ and $\tan\phi'_2$. Typical ranges of v_{C_u} and $v_{\phi'}$ as reported for example by Lee et al. (1983), Lacasse and Nadim (1996) and Lumb (1974) are $0.05 \sim 0.5$ and $0.02 \sim 0.56$, respectively. This paper assumes a coefficient of variation of 0.3 for all random variables first and then increases it to 0.7 to investigate its influences. Some investigators (e.g., Rackwitz, 2000) have suggested that the correlation between c' and ϕ' is around -0.5. Since a negative correlation between c' and ϕ' means a low c' comes with a high ϕ' and vice verse, it leads to lower p_f estimates than zero and positive correlations (e.g., Griffiths et al., 2009). For simplicity, no correlation between c' and ϕ' was considered in this paper.

The lognormally distributed undrained shear strength C_{u1} has two parameters: the mean $\mu_{C_{u1}}$ and the standard deviation $\sigma_{C_{u1}}$. The variability of C_{u1} can conveniently be expressed by the dimensionless coefficient of variation, defined as

$$v_{C_{u1}} = \frac{\sigma_{C_{u1}}}{\mu_{C_{u1}}} \tag{5}$$

The parameters of the normal distribution (of the logarithm of C_{u1}) can be obtained from the standard deviation and mean of C_{u1} as follows (e.g., Fenton and Griffiths, 2008):

$$\sigma_{\ln C_{u1}} = \sqrt{\ln\{1 + v_{C_{u1}}^2\}} \tag{6}$$

$$\mu_{\ln C_{u1}} = \ln \mu_{C_{u1}} - \frac{1}{2}\sigma_{\ln C_{u1}}^2 \tag{7}$$

Inverting Eqs. (6) and (7) gives the mean and standard deviation of C_{u1}:

$$\mu_{C_{u1}} = \exp\left(\mu_{\ln C_{u1}} + \frac{1}{2}\sigma_{\ln C_{u1}}^2\right) \tag{8}$$

$$\sigma_{C_{u1}} = \mu_{C_{u1}} = \sqrt{\exp(\sigma_{\ln C_{u1}}^2) - 1} \tag{9}$$

FIRST ORDER RELIABILITY METHOD

The FORM is a process which can be used to estimate the probability of the failure of systems involving multiple random variables with given probability density functions, in relation to a "limit state" function that separates the failure domain from the safe domain. Xu and Low (2006) used FORM combined with the finite element method to estimate the probability of failure of slopes. The conventional FORM based on the Hasofer-Lind reliability index (Hasofer and Lind, 1974), β_{HL}, obtains the reliability index, which is related to the minimum distance, in standard deviation units, between the mean values and the limit state surface. The conceptual and implementation barriers surrounding the use of β_{HL} for correlated normals and the FORM for correlated non-normals can largely be overcome, as was shown by Low and Tang (1997, 2004). Calculating the reliability index involves an iterative optimization process, in which the minimum value of a matrix calculation is found, subject to the constraint that the values are on the limit state surface. Commonly used software packages (e.g., Excel and Matlab) are easily adapted to perform the optimization (*see* e.g., www.mines.edu/~vgriffit/FORM). Once the reliability index β (the distance between the means and the closest failure point in standard deviation units) has been determined, the method assumes a "first order" limit state function tangent to the β contour, and the probability of failure, p_f follows from

$$p_f = 1 - \Phi(\beta) \tag{10}$$

If dealing with two random variables, the "first order" assumption results in a straight line limit state function, in which case p_f is the volume under the bi-variate probability density function on the failure side of the line. A similar concept applies to cases involving multiple random variables.

An advantage of the Hasofer-Lind index β_{HL} for correlated normal variates and the FORM index β for correlated non-normal variates is that the result it gives is not affected by the way the limit state function is set up. For example, the limit state function could be defined as the resistance minus the load, the factor of safety minus one, the logarithm of the factor of safety or some other algebraic combination, without influencing the computed value of β_{HL} or β.

The limit state function can sometimes be determined directly from theory, or for more complex systems, RSM needs to be used. The basic idea of the RSM is to approximate the limit state boundary by an explicit function of the random variables and to improve the approximation via iterations. For complex systems in which, for example the number of random variables exceeds thirty, RSM lacks robustness and accuracy, in which case the Monte Carlo Simulation is considered the most reasonable method.

At a detailed level, the determination of β in FORM is an iterative process (as explained by Haldar and Mahadevan, 2000; for example). An alternative interpretation involving an equivalent hyperellipsoid was given in Low and Tang (2004) and Low (2005) as follows:

$$\beta = \min_{g=0} \sqrt{\left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}^T [R]^{-1} \left\{\frac{X_i - \mu_i^N}{\sigma_i^N}\right\}} \qquad i = 1, 2, \ldots, n \tag{11}$$

where X_i is the i^{th} random variable, μ_i^N is the equivalent normal mean of the i^{th} random variable, σ_i^N is the equivalent normal standard deviation of the i^{th} random variable, $\{(X_i - \mu_i^N)/\sigma_i^N\}$ is the vector of n random variables reduced to standard normal space and $[R]$ is the correlation matrix.

For most slope stability analyses, no analytical equation exists which can serve as a limit state function. The Response Surface Method has been introduced in this study. This can be accomplished, for example, by fitting a curve to the results from several finite element analyses using the strength reduction method (e.g., Griffiths and Lane, 1999).

For example, a two-layer undrained slope with a foundation has two ($n = 2$) random variables C_{u1} (embankment) and C_{u2} (foundation). A quadratic surface without cross-terms with five ($2n + 1 = 5$) constants of the form

$$FS(\ln C_{u1}, \ln C_{u2}) = a_1 + a_2 \ln C_{u1} + a_3 \ln C_{u2} + a_4(\ln C_{u1})^2 + a_5(\ln C_{u2})^2 \qquad (12)$$

could be used to approximate the factor of safety function.

The limit state function could then be defined as the factor of safety function minus one, thus

$$g(\ln C_{u1}, \ln C_{u2}) = FS(\ln C_{u1}, \ln C_{u2}) - 1 \qquad (13)$$

In order to find the five constants in Eq. (12), five finite element analyses were run. For each random variable, its equivalent normal mean value, μ_i^N and two other values $\mu_i^N \pm m\sigma_i^N$ were sampled while fixing the other random variable at its equivalent normal mean value. Some investigators (e.g., Xu and Low, 2006; Griffiths et al., 2007) have related the two other sampling points to some factor of the standard deviation, given by m. A popular choice is $m=1$, which will be used later in this section. For cases involving a high standard deviation, the use of $m=1$ leads to some sampling points being far from the central sampling point and thus, the limit state function may not always be defined with accuracy in the zone of interest (i.e., near the tentative design point). For slope reliability analysis, however, limit state functions for slopes have been shown to be quite linear in the space of cohesion and friction angle (e.g., Mostyn and Li, 1993; Low et al., 1998), so p_f is rather insensitive to the choice of m.

Since the design point is not known in advance, the limit state function is initially derived at the equivalent normal mean which gives a first approximation of the design point. This design point can be far from the optimal one and may lead to incorrect results. The current work uses the following iteration procedure (e.g., Tandjiria et al., 2000), which leads to the limit state function being approximated at the design point.

1) Derive the limit state function at the equivalent normal mean values.
2) Use FORM to obtain the design point and hence p_f.
3) Update the limit state function using the design point just found.
4) Return to step 2) until two successive values of p_f are smaller than a prescribed tolerance.

The factor of safety at the design point should equal one at convergence.

Undrained ($\phi_u=0$) Slopes

For an undrained slope without a foundation, there is only one random variable, so p_f is simply equal to the probability that the shear strength parameter C_{u1} will be less than $C_{u1,FS=1}$, where $C_{u1,FS=1}$ is the value that results in $FS=1$. Quantitatively, this equals the area beneath the probability density function corresponding to $C_{u1} \leq C_{u1,FS=1}$. For the slope shown in Fig. 1(a), $C_{u1,FS=1}=0.122$ and $C_{u1,FS=1.25}=0.153$, so if we let $\mu_{C_{u1}}=0.153$ and $\sigma_{C_{u1}}=0.046$ ($v_{C_{u1}}=0.3$), Eqs. (6) and (7) give that the mean and standard deviation of the underlying normal distribution are $\mu_{\ln C_{u1}}=-1.920$ and $\sigma_{\ln C_{u1}}=0.294$ respectively. The probability of failure is therefore given by:

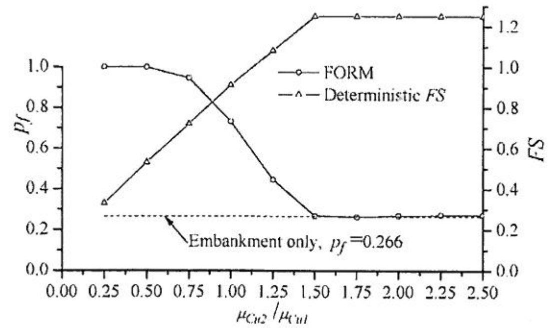

Fig. 7. Influence of $\mu_{C_{u2}}/\mu_{C_{u1}}$ on the p_f of undrained slopes by FORM ($\mu_{C_{u1}}=0.153$ and $v_{C_{u1}}=v_{C_{u2}}=0.3$)

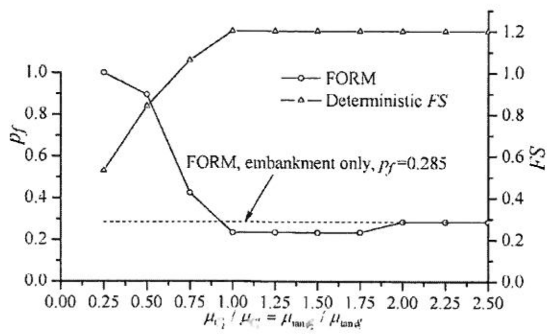

Fig. 8. Influence of $\mu_{C_2'}/\mu_{C_1'}=\mu_{\tan\phi_2'}/\mu_{\tan\phi_1'}$ on the p_f of drained slopes by FORM ($\mu_{C_1'}=0.035$, $\mu_{\tan\phi_1'}=0.364$ and $v_{C_1'}=v_{C_2'}=v_{\tan\phi_1'}=v_{\tan\phi_2'}=0.3$)

$$p_f = p[C_{u1} < 0.122] = \Phi\left(\frac{\ln 0.122 - \mu_{\ln C_{u1}}}{\sigma_{\ln C_{u1}}}\right) = 0.266 \qquad (14)$$

For the undrained two-layer slope ($\phi_u=0$) as shown in Fig. 2(a), the previously described FORM method (ignoring spatial variability) was used to calculate p_f. By changing $\mu_{C_{u2}}/\mu_{C_{u1}}$ in the range of $\{0.25, 0.5, \ldots, 2.5\}$ and fixed $\mu_{C_{u1}}=0.153$ and $v_{C_{u2}}=v_{C_{u1}}=0.3$, the influence of the strength of the foundation on the p_f was investigated, with the results shown in Fig. 7.

Drained ($C'-\tan\phi'$) Slopes

For the drained slope without foundation as shown in Fig. 4(a), the shear strength parameters C_1' and $\tan\phi_1'$ are assumed to be random variables and the same method that was used previously for the undrained slope ($\phi_u=0$) with a foundation was used to calculate the p_f. For $\mu_{C_1'}=0.035$, $\mu_{\tan\phi_1'}=0.364$ and $v_{C_1'}=v_{\tan\phi_1'}=0.3$, the p_f was found to be 0.285.

For the drained two-layer slope shown in Fig. 5(a), the shear strength parameters C_1', C_2', $\tan\phi_1'$ and $\tan\phi_2'$ are assumed to be random variables. The following quadratic surface without cross-terms is used to approximate the factor of safety function.

$$FS(\ln C_1', \ln C_2', \ln(\tan\phi_1'), \ln(\tan\phi_2'))$$
$$= a_1 + a_2 \ln C_1' + a_3 \ln C_2' + a_4 \ln(\tan\phi_1') + a_5 \ln(\tan\phi_2') + a_6(\ln C_1')^2 + a_7(\ln C_2')^2 + a_8(\ln(\tan\phi_1'))^2 + a_9(\ln(\tan\phi_2'))^2 \quad (15)$$

By changing $\mu_{C_2'}/\mu_{C_1'} = \mu_{\tan\phi_2'}/\mu_{\tan\phi_1'}$ in the range of {0.25, 0.5, . . . , 2.5} and fixing $\mu_{C_1'} = 0.035$, $\mu_{\tan\phi_1'} = 0.364$ and $v_{C_1'} = v_{C_2'} = v_{\tan\phi_1'} = v_{\tan\phi_2'} = 0.3$, the previously described procedure was used once more to calculate the p_f. The results are shown in Fig. 8. For the case when $\mu_{C_2'}/\mu_{C_1'} = \mu_{\tan\phi_2'}/\mu_{\tan\phi_1'} = 1.0$, the two-layer slope can be treated as one-layer slope modeled with statistical strength parameters $\mu_{C_1'} = 0.035$, $\mu_{\tan\phi_1'} = 0.364$ and $v_{C_1'} = v_{\tan\phi_1'} = 0.3$. If a limit state function similar to Eq. (12) was used, the corresponding p_f was found to be 0.262.

RANDOM FINITE ELEMENT METHOD

In this section, the results of full nonlinear RFEM analyses with Monte-Carlo simulations are compared with results from FORM.

The RFEM involves the generation and mapping of a random field of properties onto a finite element mesh. The current on-line RFEM codes have implemented only normal, lognormal and bounded distributions (Fenton and Griffiths, 2008). There is no restriction, however, on the type of distribution that could be modeled by the RFEM, but a normal transformation is available (e.g., Fig. 5 in Low and Tang, 2007). Since the random field in RFEM is generated in the underlying normal space, it is easy to map this normal distribution to some other distribution types. Full account is taken of local averaging and variance reduction (Fenton and Vanmarcke, 1990) over each element, and an exponentially decaying (Markov) spatial correlation function is incorporated. The random field is initially generated and properties are assigned to the elements. A typical elastoplastic finite element analysis follows (*see* e.g., Griffiths and Lane, 1999). Failure of any particular simulation was determined on the basis of two criteria: 1) Failure of the algorithm to converge within a user-specified iteration ceiling (typically set to 500), or 2) The observation of a sudden increase in nodal displacements due to the inability of the algorithm to find a stress distribution that satisfies both Mohr-Coulomb's failure criterion and global equilibrium with the gravity loads. The convergence criterion was based on a comparison of successive self-equilibrating "bodyload" vectors. If the absolute change from one iteration to the next of all the components of the "bodyload" vector, non-dimensionalized with respect to the component of largest absolute magnitude, falls below a tolerance level of 0.0001, converge is said to have occurred, and the slope is deemed not to have failed (*see* Smith and Griffiths, 2004, for more details). During the Monte-Carlo simulations, the location of the failure surface is itself a random process. While this is a worthy topic for further investigation, the authors' experience from inspecting individual simulations that lead to failure is that the kinematics of the problem tend to favor global mechanisms. The analysis is repeated numerous times using Monte-Carlo simulations. Each realization of the Monte-Carlo process involves the same mean, standard deviation and spatial correlation length of soil properties, however the spatial distribution of properties varies from one realization to the next. Following a "sufficient" number of realizations, the p_f can be easily estimated by dividing the number of failures by the total number of simulations. By increasing gradually the number of realizations, it was determined that 2000 realizations of the Monte-Carlo process for each parametric group were sufficient to give reliable and reproducible estimates of p_f. The analysis has the option of including cross correlation between properties and anisotropic spatial correlation lengths (e.g., the spatial correlation length in a naturally occurring stratum of soil is often higher in the horizontal direction). Since the actual undrained shear strength field is lognormally distributed, its logarithm yields an "underlying" normally distributed (or Gaussian) field. The spatial correlation length is measured with respect to this underlying field. The spatial correlation length (e.g., $\theta_{\ln C_{u1}}$) describes the distance over which the spatially random values will tend to be significantly correlated in the underlying Gaussian field. Thus, a large value of $\theta_{\ln C_{u1}}$ will imply a smoothly varying field, while a small value will imply a ragged field.

In this work, an exponentially decaying (Markovian) correlation function is used of the form, for example:

$$\rho(\tau) = e^{-\frac{2\tau}{\theta_{\ln C_u}}} \quad (16)$$

where $\rho(\tau)$ is the correlation coefficient between properties assigned to two points in the random field separated by an absolute distance τ.

In the current study, the spatial correlation length has been non-dimensionalized by dividing it by the height of the embankment H and will be expressed in the form, for example:

$$\Theta_{C_{u1}} = \theta_{\ln C_{u1}}/H \quad (17)$$

In order to study the p_f of layered slopes, the RFEM was further developed to have the ability to simulate multiple random fields. Figure 9 shows a two-layer slope where each layer is modeled with the same mean and standard deviation, but different spatial correlation lengths. A relatively low spatial correlation length of $\Theta_{C_{u1}} = 0.2$ was assigned to the embankment and a relatively high spatial correlation length of $\Theta_{C_{u2}} = 2.0$ to the foundation. A ten times difference was chosen to show different spatial correlation lengths. The figure depicts the variations of c_{u1} and c_{u2}, and has been scaled in such a way that dark and light regions depict "strong" and "weak" soil, respectively.

The input parameters relating to the mean, standard deviation and spatial correlation length are assumed to be defined at the "point" level. While statistics at this resolution are obviously impossible to measure in practice, they represent a fundamental baseline of the inherent soil variability which can be corrected through local averaging to take account of the sample size. In the context of

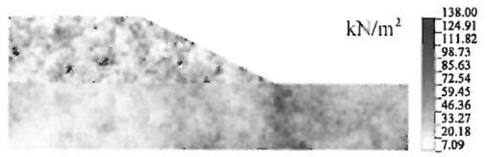

Fig. 9. Undrained cohesion portrayal of different spatial correlation length in the embankment and foundation in RFEM analysis ($\mu_{C_{u1}} = \mu_{C_{u2}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.3$, $\Theta_{C_{u1}} = 0.2$ and $\Theta_{C_{u2}} = 2.0$)

the RFEM approach, each finite element is assigned a constant property. The "sample" is represented by the size of each finite element used to discretize the slope. If the point distribution is normal, local averaging results in a reduced variance but the mean is unaffected. In a lognormal distribution, however, both the mean and the standard deviation are reduced by local averaging. Following local averaging, the adjusted statistics ($\mu_{C_{u1A}}$, $\sigma_{C_{u1A}}$) represent the mean and standard deviation of the lognormal field that is actually mapped onto the finite element mesh. Further details of RFEM can be found in Griffiths and Fenton (2004) and Fenton and Griffiths (2008).

Undrained ($\phi_u = 0$) *Slopes*

By changing $\mu_{C_{u2}}/\mu_{C_{u1}}$ in the range of {0.25, 0.5, . . . , 2.5} and fixing $\mu_{C_{u1}} = 0.153$, $v_{C_{u2}} = v_{C_{u1}} = 0.3$ and $\Theta_{\ln C_{u2}} = \Theta_{\ln C_{u1}} = 0.5$, the RFEM was used to calculate the p_f of the undrained two-layer slopes as shown in Fig. 2(a). The results are shown in Fig. 10. Also plotted in Fig. 10 is the "embankment only" result $p_f = 0.071$ which is for the slope shown in Fig. 1(a) by treating C_{u1} as a random variable with statistical strength parameters $\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = 0.3$ and $\Theta_{\ln C_{u1}} = 0.5$.

The foundation strength has little influence on the p_f of two-layer slopes if $\mu_{C_{u2}}/\mu_{C_{u1}} > 1.50$ for both the RFEM and FORM (ignoring spatial variability). The results when $\mu_{C_{u2}}/\mu_{C_{u1}} = 1.50$ are the most interesting in that two mechanisms are trying to form at the same time, as can be seen in Fig. 2(b). The RFEM successfully gave a higher p_f of 0.118 for the two-layer slope than the p_f of 0.071 in the "embankment only" case which has one mechanism as shown in Fig. 1(b). In other words, the RFEM accurately predicts the system probability of failure, but FORM (ignoring spatial variability) only catches the failure mechanism with the highest p_f. This phenomenon was further investigated by varying $\Theta_{\ln C_{u2}} = \Theta_{\ln C_{u1}}$ in the range of {0.125, 0.25, . . . , 32.0} while fixing $\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.3$ and $\mu_{C_{u2}}/\mu_{C_{u1}} = 1.5$. The results are shown in Fig. 11. It can be seen that the maximum difference between p_f of the two-layer slope (system probability of failure) and the p_f of "embankment only" occurs at $1.0 \le \Theta_{\ln C_{u2}} = \Theta_{\ln C_{u1}} \le 2.0$. When $\Theta_{\ln C_{u2}} = \Theta_{\ln C_{u1}} = 1.0$, the system p_f is 40% higher than the p_f of "embankment only". It is noted that the p_f of the two-layer slope (system probability of failure) and the p_f of "embankment only" tend to be the same when spatial correlation lengths tend to infinite. The result indicates that if the mean strength of the foundation is much higher than that

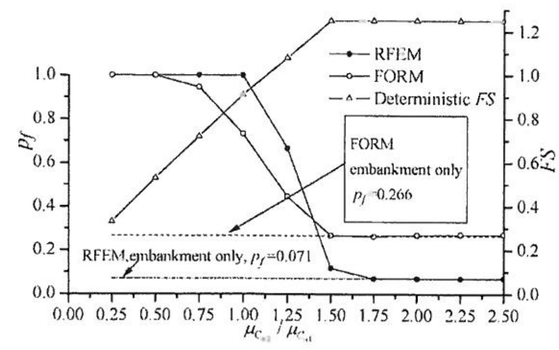

Fig. 10. Influence of $\mu_{C_{u2}}/\mu_{C_{u1}}$ on the p_f of undrained slopes by RFEM ($\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.3$ and $\Theta_{\ln C_{u1}} = \Theta_{\ln C_{u2}} = 0.5$)

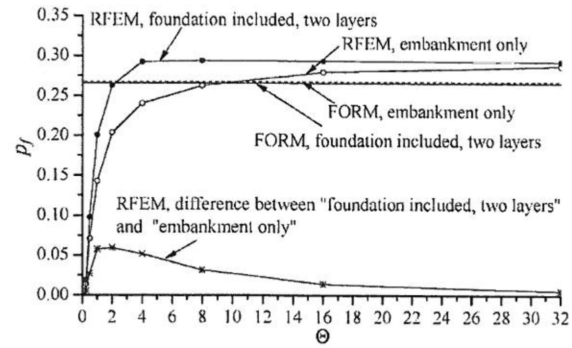

Fig. 11. Influence of spatial correlation length on the p_f of undrained slopes by RFEM ($\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.3$ and $\mu_{C_{u2}}/\mu_{C_{u1}} = 1.5$)

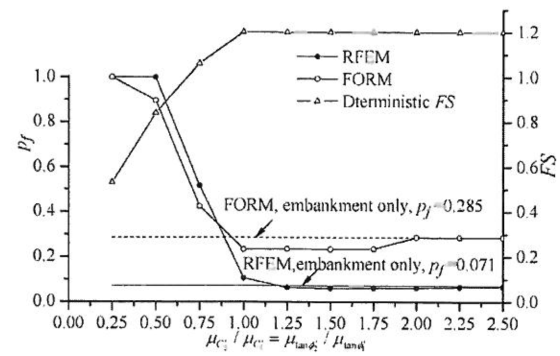

Fig. 12. Influence of $\mu_{C'_2}/\mu_{C'_1} = \mu_{\tan\phi'_2}/\mu_{\tan\phi'_1}$ on the p_f of drained slopes by RFEM ($\mu_{C'_1} = 0.035$, $\mu_{\tan\phi'_1} = 0.364$, $v_{C'_2} = v_{C'_1} = v_{\tan\phi'_2} = v_{\tan\phi'_1} = 0.3$ and $\Theta_{C'_2}/\Theta_{C'_1} = \Theta_{\tan\phi'_2} = \Theta_{\tan\phi'_1} = 0.5$)

of the embankment ($\mu_{C_{u2}}/\mu_{C_{u1}} = 1.5$ in this case), the variability of the strength of foundation has little influence on the system reliability when spatial correlation lengths tend to infinity.

Drained ($C' - \tan\phi'$) *Slopes*

By changing $\mu_{C'_2}/\mu_{C'_1} = \mu_{\tan\phi'_2}/\mu_{\tan\phi'_1}$ in the range of {0.25, 0.5, . . . , 2.5} while fixing $\mu_{C'_1} = 0.035$, $\mu_{\tan\phi'_1} = 0.364$, $v_{C'_2} = v_{C'_1} = v_{\tan\phi'_2} = v_{\tan\phi'_1} = 0.3$ and $\Theta_{C'_2} = \Theta_{C'_1} = \Theta_{\tan\phi'_2} = \Theta_{\tan\phi'_1} =$

0.5, the RFEM was used to calculate the p_f of the drained two-layer slope shown in Fig. 5(a) with the results being shown in Fig. 12. Also plotted in Fig. 12 is the "embankment only" result $p_f = 0.071$, which is for the slope shown in Fig. 4(a) after treating C_i' and $\tan\phi_i'$ as random variables with the same statistical parameters.

The foundation strength has little influence on the p_f of the two-layer slopes if $\mu_{C_f'}/\mu_{C_i'} = \mu_{\tan\phi_f'}/\mu_{\tan\phi_i'} > 1.0$ for both RFEM and FORM. When $\mu_{C_f'}/\mu_{C_i'} = \mu_{\tan\phi_f'}/\mu_{\tan\phi_i'} = 1.0$. The RFEM successfully gave a higher p_f of 0.105 for the two-layer slope than the p_f of 0.071 of the "embankment only" which has one mechanism as shown in Fig. 4(b). In other words, RFEM accurately predicts the system probability of failure, but FORM (ignoring spatial variability) only catches the failure mechanism with the highest p_f. A great benefit of the RFEM is that the shape and location of the failure surface is not determined a priori and is able to "seek out" the most critical path through the heterogeneous soil mass (e.g., Griffiths et al., 2006). This merit enables the RFEM to catch every failure in a suite of Monte-Carlo simulations. By ignoring spatial correlation, FORM assumes all slip surfaces are perfect correlated (Oka and Wu, 1990; Chowdhury and Xu, 1995) and thus can catch only the smallest reliability index. This phenomenon was further investigated by varying $\Theta_{C_f'} = \Theta_{C_i'} = \Theta_{\tan\phi_f'} = \Theta_{\tan\phi_i'}$ in the range of {0.125, 0.25, . . . , 32.0} while fixing $\mu_{C_i'} = 0.035$, $\mu_{\tan\phi_i'} = 0.364$, $v_{C_f'} = v_{C_i'} = v_{\tan\phi_f'} = v_{\tan\phi_i'} = 0.3$ and, with the results shown in Fig. 13. Also plotted in Fig. 13 are the results after treating the slope as a one-layer slope modeled by one random field with statistical parameters $\mu_{C_i'} = 0.035$, $\mu_{\tan\phi_i'} = 0.364$, $v_{C_i'} = v_{\tan\phi_i'} = 0.3$ and $\Theta_{C_i'} = \Theta_{\tan\phi_i'} = 0.5$. The p_f results from the one-layer slopes are always higher than the p_f of "embankment only," but always lower than the p_f of the two-layer slopes. The result indicates that if the statistical strength of the foundation is the same as the embankment, the variability of the strength of foundation has significant influence on the system reliability. Introducing the second random field of the same statistical strength parameters for the foundations will increase the system probability of failure even when spatial correlation lengths tend to infinity for drained slopes.

It was also noted that FORM (assuming an infinite spatial correlation) gave a reverse trend compared to RFEM in which the p_f of the two-layer slope is the lowest and the p_f of the "embankment only" is the highest. The p_f of a one-layer slope with a foundation takes an intermediate value. It should be mentioned that a one-layer slope and "embankment only" both have only two random variables with same statistical properties. FORM combined with RSM should give the same p_f in both cases with no difference due to numerical rounding. For two-layer slopes, since more variances were introduced by including the foundation, FORM combined with RSM should give a higher system p_f. However, FORM failed to catch the influence of the variability of the strengths of the foundation on the system reliability.

Numerical Results When $v = 0.7$

All previous results used $v = 0.3$. In this section we have increased the variance to $v = 0.7$. Figures 14 and 15 show results of undrained slopes and Figs. 16 and 17 show results of drained slopes.

For results of undrained slopes using the RFEM, a comparison of Figs. 14 to 10 indicates that in terms of the

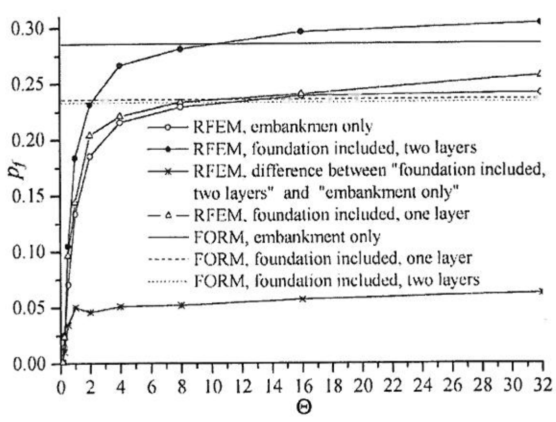

Fig. 13. Influence of spatial correlation length on the p_f of drained slopes by RFEM ($\mu_{C_i'} = 0.035$, $\mu_{\tan\phi_i'} = 0.364$, $v_{C_f'} = v_{C_i'} = v_{\tan\phi_f'} = v_{\tan\phi_i'} = 0.3$ and $\mu_{C_f'}/\mu_{C_i'} = \mu_{\tan\phi_f'}/\mu_{\tan\phi_i'} = 1.0$)

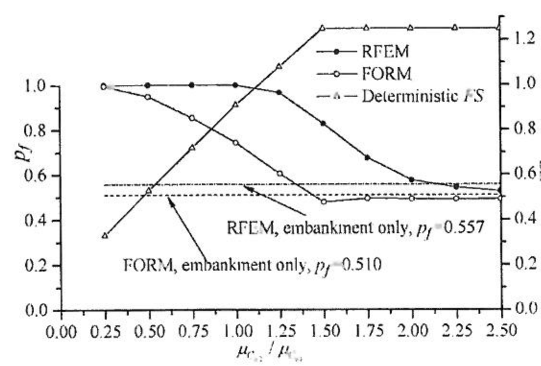

Fig. 14. Influence of $\mu_{C_{u2}}/\mu_{C_{u1}}$ on the p_f of undrained slopes by RFEM ($\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.7$ and $\Theta_{\ln C_{u1}} = \Theta_{\ln C_{u2}} = 0.5$)

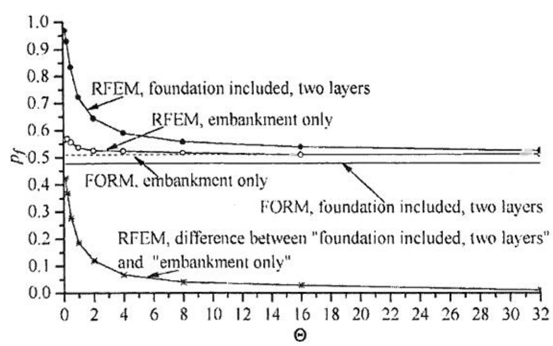

Fig. 15. Influence of spatial correlation length on the p_f of undrained slopes by RFEM ($\mu_{C_{u1}} = 0.153$, $v_{C_{u1}} = v_{C_{u2}} = 0.7$ and $\mu_{C_{u2}}/\mu_{C_{u1}} = 1.5$)

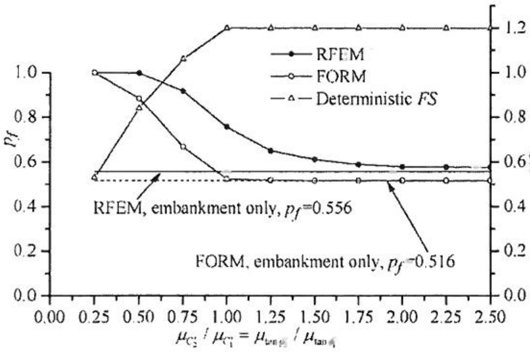

Fig. 16. Influence of $\mu_{C_2'}/\mu_{C_1'}=\mu_{\tan\phi_2'}/\mu_{\tan\phi_1'}$ on the p_f of drained slopes by RFEM ($\mu_{C_1'}=0.035$, $\mu_{\tan\phi_1'}=0.364$, $v_{C_1'}=v_{C_2'}=v_{\tan\phi_1'}=v_{\tan\phi_2'}=0.7$ and $\Theta_{C_1'}=\Theta_{C_2'}=\Theta_{\tan\phi_1'}=\Theta_{\tan\phi_2'}=0.5$)

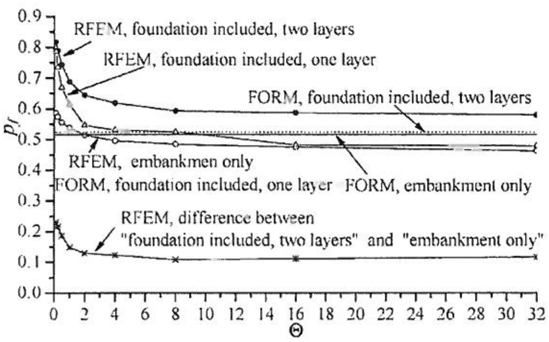

Fig. 17. Influence of spatial correlation length on the p_f of drained slopes by RFEM ($\mu_{C_1'}=0.035$, $\mu_{\tan\phi_1'}=0.364$, $v_{C_1'}=v_{C_2'}=v_{\tan\phi_1'}=v_{\tan\phi_2'}=0.7$ and $\mu_{C_2'}/\mu_{C_1'}=\mu_{\tan\phi_2'}/\mu_{\tan\phi_1'}=1.0$)

strength ratio, the p_f of the two-layer slope ceases to be higher than the p_f of "embankment only", which increases from $\mu_{C_{u2}}/\mu_{C_{u1}}=1.5$ when $v=0.3$ to $\mu_{C_{u2}}/\mu_{C_{u1}}=2.0$ when $v=0.7$. For results of drained slopes using the RFEM, a comparison of Figs. 16 to 12 indicates that the p_f of the two-layer slope is always higher than the p_f of "embankment only" when $v=0.7$.

A comparison of the results of the REFM in Figs. 15 to 11 and Figs. 17 to 13 shows that the difference between p_f of two-layer slopes (system probability of failure) and the p_f of "embankment only" increases as v increases.

When FORM was used, including the foundation made no substantial difference to p_f in any case when $v=0.7$. The results confirmed again that FORM failed to catch the influence of the variability of the foundation on the system reliability.

It should be mentioned that the p_f when using the RFEM decreased as spatial correlation lengths increased when $v=0.7$, but the p_f increased as spatial correlation lengths increased when $v=0.3$. This phenomenon has been well explained by Griffiths and Fenton (2004).

CONCLUDING REMARKS

This paper used the probabilistic method (FORM) and the simulation method (RFEM) to study the probability of the failure of slopes. Numerical results showed that the RFEM can accurately predict the system probability of failure of slopes in the framework of Monte-Carlo simulations. However, FORM, which targets the minimum reliability index related to a particular slip surface, cannot give accurate information regarding the system reliability of slopes.

ACKNOWLEDGEMENT

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NOTATION

- c_1': drained cohesion of the embankment
- c_2': drained cohesion of the foundation
- c_{u1}: undrained cohesion of the embankment
- c_{u2}: undrained cohesion of the foundation
- C_1': dimensionless drained cohesion of the embankment
- C_2': dimensionless drained cohesion of the foundation
- C_{u1}: dimensionless undrained cohesion of the embankment
- C_{u2}: dimensionless undrained cohesion of the foundation
- $C_{u1,FS=1}$: dimensionless undrained cohesion when $FS=1.0$
- $C_{u1,FS=1.25}$: dimensionless undrained cohesion when $FS=1.25$
- D: foundation depth ratio
- f: joint probability density function
- g: limit state function
- FS: factor of safety
- H: slope height
- m: constant used for sampling limit state function
- n: number of random variables
- p_f: probability of failure
- $[R]$: correlation matrix
- X: random variables
- X_i: the i^{th} random variable
- α: slope angle
- β: FORM reliability index
- β_g: the generalized reliability index
- β_{HL}: the Hasofer-Lind reliability index
- β_{min}: the minimum reliability index
- γ: soil unit weight
- γ_{sat}: saturated soil unit weight
- $\theta_{\ln C_{u1}}$: spatial correlation length of undrained cohesion of the embankment
- Θ: dimensionless spatial correlation length
- $\Theta_{C_{u1}}$: dimensionless spatial correlation length of undrained cohesion of the embankment

$\Theta_{C_{u2}}$: dimensionless spatial correlation length of undrained cohesion of the foundation
$\Theta_{C'_1}$: dimensionless spatial correlation length of drained cohesion of the embankment
$\Theta_{C'_2}$: dimensionless spatial correlation length of drained cohesion of the foundation
$\Theta_{\tan\phi'_1}$: dimensionless spatial correlation length of drained tangent friction angle of the embankment
$\Theta_{\tan\phi'_2}$: dimensionless spatial correlation length of drained tangent friction angle of the foundation
$\mu_{C_{u1}}$: mean dimensionless undrained cohesion of the embankment
$\mu_{C_{u2}}$: mean dimensionless undrained cohesion of the foundation
$\mu_{C_{u1A}}$: mean dimensionless undrained cohesion after local averaging of the embankment
$\mu_{C'_1}$: mean dimensionless drained cohesion of the embankment
$\mu_{C'_2}$: mean dimensionless drained cohesion of the foundation
$\mu_{\ln C_{u1}}$: equivalent normal mean of dimensionless undrained cohesion of the embankment
μ_i^N: equivalent normal mean of the i^{th} random variable
$\mu_{\tan\phi'_1}$: mean drained tangent friction angle of the embankment
$\mu_{\tan\phi'_2}$: mean drained tangent friction angle of the foundation
v: coefficient of variation
$v_{C_{u1}}$: coefficient of variation of dimensionless undrained cohesion of the embankment
$v_{C_{u2}}$: coefficient of variation of dimensionless undrained cohesion of the foundation
$v_{C'_1}$: coefficient of variation of dimensionless drained cohesion of the embankment
$v_{C'_2}$: coefficient of variation of dimensionless drained cohesion of the foundation
$v_{\tan\phi'_1}$: coefficient of variation of tangent drained friction angle of the embankment
$v_{\tan\phi'_2}$: coefficient of variation of tangent drained friction angle of the foundation
ρ: cross correlation coefficient
$\rho(\tau)$: correlation coefficient between properties assigned to two points
$\sigma_{C_{u1}}$: standard deviation of dimensionless undrained cohesion of the embankment
$\sigma_{C_{u1A}}$: standard deviation of dimensionless undrained cohesion after local averaging of the embankment
$\sigma_{\ln C_{u1}}$: equivalent normal standard deviation of undrained cohesion of the embankment
σ_i^N: equivalent normal standard deviation of the i^{th} random variable
τ: absolute distance between two points in a random field
ϕ_u: undrained friction angle
ϕ'_1: drained friction angle of the embankment
ϕ'_2: drained friction angle of the foundation
$\Phi(\cdot)$: the cumulative standard normal distribution function.

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