Geotechnical resistance factors for ultimate limit state design of deep foundations in frictional soils

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Abstract: This paper investigates the probabilistic nature of ultimate limit state failures of deep foundations in purely frictional soils (e.g., sands). In so doing, the theory required to predict both the probability of ultimate limit state failure and the resistance factors needed to avoid this limit state are proposed. The proposed resistance factors are functions of site understanding and failure consequence, and the theory leading to these resistance factors is validated via Monte Carlo simulation of a two-dimensional spatially variable random field. In both the theory and the simulation, a pile is assumed to be placed vertically at a certain position in the soil mass, and the soil is sampled at various distances from the pile to come up with characteristic soil properties (namely friction angle) for use in the pile design. Agreement between theory and simulation is found to be very good. The theoretical model is then employed to determine upper bound geotechnical resistance factors, which can be used to complement current ultimate limit state design code calibration efforts. An example of such a calibration is presented.

Key words: reliability-based design, ultimate limit state design, load and resistance factor design, deep foundations, geotechnical resistance factor.

Résumé : Cet article investigue la nature probabiliste des ruptures à l'état limite ultime de fondations profondes dans des sols purement frictionnels (ex. sables). Pour ce faire, la théorie requise pour prédire la rupture à l'état limite ultime ainsi que les facteurs de résistance nécessaires pour éviter cet état limite sont proposés. Les facteurs de résistance proposés sont fonction de la compréhension du site et des conséquences de la rupture, et la théorie qui supporte ces facteurs est validée par une simulation Monte Carlo d'un champ aléatoire ayant des variations spatiales en deux dimensions. Autant dans la théorie que dans la pratique, on suppose que le pieu est placé verticalement à une certaine position dans la masse de sol, et le sol est échantillonné à plusieurs distances du pieu afin d'obtenir les propriétés caractéristiques du sol (comme l'angle de friction) pour la conception du pieu. La concordance entre la théorie et la simulation est très bonne. Le modèle théorique est ensuite utilisé pour déterminer les facteurs de résistance géotechnique de la frontière supérieure qui peuvent être utilisés pour complémenter les efforts actuels de calibrage du code de conception à l'état limite ultime. Un exemple d'un tel calibrage est présenté.

Mots-clés : conception basée sur la fiabilité, conception à l'état limite ultime, conception des facteurs de charge et résistance, fondations profondes, facteur de résistance géotechnique.

[Traduit par la Rédaction]

Introduction

The soil supports a pile foundation through a combination of end bearing and side friction and (or) cohesion. This paper only examines axial frictional resistance of piles, as is found in frictional soils, and end bearing is ignored. A mathematical theory is developed to theoretically estimate the failure probability of deep foundations in soils under such effective stress conditions. The theory is validated by simulation and then used to estimate failure probabilities and resistance factors required for design. The resulting recommended design factors can then be used to complement existing load and resistance factor design (LRFD) calibration efforts for deep foundations, such as the extensive experimental work carried out by Paikowsky (2004).

The ultimate axial resistance of a pile, R_u, due to frictional resistance between the pile and its surrounding soil is given by

$$[1] \qquad R_u = \int_0^H p\tau(z)\, dz$$

where p is the effective perimeter length of the pile section, $\tau(z)$ is the average ultimate shear stress acting on the perimeter of the pile at depth z, and H is the buried length of the pile. The average ultimate shear stress in soils under effective stress conditions can be estimated by

$$[2] \qquad \tau(z) = K(z)\sigma'_o(z)\tan\delta(z)$$

where $K(z)$ is the coefficient of lateral earth pressure at depth z, $\sigma'_o(z)$ is the effective vertical stress at depth z, and $\delta(z)$ is the average interface friction angle between the soil and the pile perimeter at depth z. The effective vertical stress, $\sigma'_o(z)$, and interface friction angle, $\delta(z)$, can be written as

Received 31 March 2011. Accepted 19 July 2011. Published at www.nrcresearchpress.com/cgj on 2 November 2011.

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$$[3a] \qquad \sigma'_{o}(z) = \gamma z$$

$$[3b] \qquad \delta(z) = b\phi(z)$$

where γ is the effective unit weight of soil, b is a reduction factor, which is commonly in the range 0.5–0.8 (Das 2000), and $\phi(z)$ is the average effective angle of internal friction of the soil around the pile perimeter at depth z.

The value of $K(z)$ is influenced by the material, size, shape of the pile, and the method of pile installation, as well as by the soil's friction angle, compressibility, and degree of overconsolidation. A reasonable approximation to $K(z)$ is generally taken to be the Rankine passive earth pressure coefficient, $K_{p}(z)$, at the top of the pile trending to somewhat less than the at-rest earth pressure coefficient, $K_{o}(z)$, at the pile tip (Das 2000). For coarse-grained soils, the at-rest earth pressure coefficient is estimated by Jaky (1944) to be

$$[4] \qquad K_{o}(z) = 1 - \sin\phi(z)$$

Table 1 presents the average values of $K(z)$ recommended by Das (2000), which are given in terms of Jaky's at-rest earth pressure coefficient.

In this paper the earth pressure coefficient is assumed to be

$$[5] \qquad K(z) = a[1 - \sin\phi(z)]$$

where a lies in the range $1 < a < 1.8$. Three different values of a, namely $a = 1.0$, 1.2, and 1.4, are considered here for bored, low-displacement, and high-displacement driven piles, respectively. These are the midpoints of the ranges given in Table 1.

If the model parameters (a, b, and effective perimeter, p) and the soil parameters (γ and $\phi(z)$), used in the above geotechnical models are all precisely measured along the pile depth, and it is further assumed that Jaky's model is correct, then the ultimate frictional resistance of a pile having length, H, will be

$$[6] \qquad R_{u} = \int_{0}^{H} p\gamma z a[1 - \sin\phi(z)] \tan b\phi(z)\, dz$$

and this will be taken to be the "true" expression of the ultimate pile resistance in both the theory and the simulation results to follow.

It is well known that all of the variables on the right-hand side (RHS) of eq. [6], including H, are actually uncertain and thus random, which means that R_{u} is also random. The task of this paper is to determine the distribution of R_{u}, so that failure probability estimates can be made and required resistance factors determined.

As will be shown in the section "Theoretical estimation of failure probability", the model parameters a, p, and γ all cancel out and disappear in the calculation of the pile failure probability. The model parameter b also largely cancels out (to first order). Thus, the precise values of these parameters do not influence the resistance factors determined here, and only the friction angle, $\phi(z)$, needs to be explicitly modeled as a spatially varying random field in this analysis. This does not mean that these model parameters do not influence the true failure probability, just that this study concentrates on failure probabilities arising due to uncertainties in the loads and in the friction angle and not due to uncertainties in these model parameters. Uncertainty in these parameters can, however, be handled indirectly by suitably adjusting the coefficient of variation of the friction angle. This issue will be discussed at further length in the section "Conclusions".

The limit state design (LSD) framework basically involves identifying possible failure modes (e.g., punching shear failure, or excessive settlement) and then ensuring that the factored geotechnical resistance at each limit state is not less than the factored load. At the ultimate limit state under consideration here, the design requirement is

$$[7] \qquad \varphi_{gu}\hat{R}_{u} \geq \sum_{i} I_{i}\alpha_{i}\hat{F}_{i}$$

where φ_{gu} is the ultimate geotechnical resistance factor, $\hat{R}_{u}$ is the characteristic (design) ultimate geotechnical resistance, I_{i} is an importance factor corresponding to the ith characteristic load effect, $\hat{F}_{i}$, and α_{i} is the ith load factor.

The characteristic ultimate geotechnical resistance, $\hat{R}_{u}$, is determined using characteristic soil properties, in this case the characteristic value of the soil's friction angle, ϕ. To obtain the characteristic soil property, the soil is assumed to be sampled over a single column somewhere in the vicinity of the pile, for example, by a cone penetration test (CPT) or standard penetration test (SPT) sounding. The sample is assumed to yield a sequence of m observed friction angle values, $\hat{\phi}_{1}, \hat{\phi}_{2}, \ldots, \hat{\phi}_{m}$. The characteristic value of the friction angle, $\hat{\phi}$, is defined in this paper as an arithmetic average of the sampled observations, $\hat{\phi}_{i}$,

$$[8] \qquad \hat{\phi} = \frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_{i}$$

The characteristic ultimate geotechnical resistance, $\hat{R}_{u}$, is then obtained by using the characteristic friction angle in eq. [6],

$$[9] \qquad \hat{R}_{u} = \frac{1}{2}pa\gamma H^{2}(1 - \sin\hat{\phi})\tan(b\hat{\phi})$$

To determine the geotechnical resistance factor, φ_{gu}, required to achieve a certain acceptable reliability, the failure probability of the pile must be estimated. This probability will depend on the load distribution, the load and resistance factors selected, and the resistance distribution. The resistance distribution is discussed in the section "Random soil model", and the load distribution is discussed in the section "Random load model". The section "Theoretical estimation of failure probability" develops the theory proposed to allow the estimation of the pile failure probability, while the section "Comparison of theoretical and simulated failure probabilities" describes the simulation used to validate the theory and compares the simulation results to those predicted theoretically.

The LRFD approach involves selecting one or more maximum acceptable failure probability levels, p_{m}. The choice of p_{m} derives from a consideration of acceptable risk and directly influences the size of φ_{gu}. In this research, four maximum acceptable failure probabilities, 10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5}, will be considered. These can be considered to be target lifetime failure probabilities, so long as the selected coeffi-

Table 1. Lateral earth pressure recommendations (Das 2000).

Pile type	$K(z)$
Bored or jetted	$\simeq K_o(z) = 1 - \sin\phi(z)$
Low-displacement driven	$\simeq K_o(z) = 1 - \sin\phi(z)$ to $1.4K_o(z) = 1.4[1 - \sin\phi(z)]$
High-displacement driven	$\simeq K_o(z) = 1 - \sin\phi(z)$ to $1.8K_o(z) = 1.8[1 - \sin\phi(z)]$

cients of variation of the resistance and load are those of the minimum lifetime resistance and maximum lifetime loads (particularly live loads), respectively. Some of these failure probabilities, i.e., 10^{-3}, 10^{-4}, and 10^{-5}, might be appropriate for designs involving low (e.g., storage facilities), medium (typical structures), and high (e.g., hospitals and schools) failure consequence structures, respectively. A reasonable value of maximum acceptable failure probability for a single driven pile within a redundant pile group may be in the range of 10^{-2}–10^{-3} (Allen 2005). The geotechnical resistance factors required to achieve these maximum acceptable failure probabilities will be recommended in the section "Geotechnical resistance factors".

Random soil model

The friction angle, ϕ, is assumed to be bounded both above and below, so that neither normal nor lognormal distributions are appropriate. While a beta distribution is often used for bounded random variables, a beta distributed random field has a complex joint distribution, and both the theoretical development and simulation become cumbersome and numerically difficult. In that the best friction angle distribution has yet to be established by the geotechnical community, the authors choose a "tanh" bounded distribution, which is very similar in its properties to the beta distribution, but which arises as a simple transformation of a standard normal random field, $G_\phi(z)$, according to

$$[10] \qquad \phi(z) = \phi_{\min} + (\phi_{\max} - \phi_{\min})\left\{1 + \tanh\left[\frac{sG_\phi(z)}{2\pi}\right]\right\}$$

where $\phi_{\min}$ and $\phi_{\max}$ are the minimum and maximum friction angles in radians, respectively, and s is a scale factor that governs the friction angle variability between its two bounds (see Fenton and Griffiths 2008, for more details). This distribution has at least three advantages over the beta distribution: (*i*) since it is derived from a standard normal random field, its theoretical properties can be determined relatively accurately (which is important when developing a theoretical model), (*ii*) its parameters can be easily obtained from site data through a suitable transformation (similar to the transformation required to determine the parameters of a lognormal distribution), and (*iii*) the simulation of the friction angle field is exact if the underlying standard normal field is exact (and most random field simulators are aimed at rendering a standard normal field accurately).

Figure 1 illustrates how the distribution of ϕ, normalized over the range $\phi_{\min} = 0$ to $\phi_{\max} = 1$, changes as its variability changes, going from an almost uniform distribution at $s = 5$ to a very normal looking distribution for smaller s. Thus, varying s between about 0.1 and 5.0 leads to a wide range in the stochastic behaviour of ϕ. In all cases, the distribution is assumed in this paper to be symmetric so that the midpoint between $\phi_{\min}$ and $\phi_{\max}$ is the mean. Values of s greater than about five lead to a U-shaped distribution (higher at the boundaries), which is deemed to be unrealistic.

The following relationship between s and the standard deviation of ϕ derives from a third-order Taylor series approximation to tanh and a first-order approximation to the final expectation (Fenton and Griffiths 2008),

$$[11] \qquad \sigma_\phi \simeq \frac{0.46(\phi_{\max} - \phi_{\min})s}{\sqrt{4\pi^2 + s^2}}$$

where $\phi_{\min}$ and $\phi_{\max}$ are in radians.

The relationship between the coefficient of variation of the friction angle, $v_\phi = \sigma_\phi/\mu_\phi$, and the parameter s used in eq. [10] can be obtained by inverting eq. [11] (see Naghibi (2010) for details)

$$[12] \qquad s \simeq \frac{2\pi v_\phi \mu_\phi}{\sqrt{(0.46)^2(\phi_{\max} - \phi_{\min})^2 - (v_\phi \mu_\phi)^2}}$$

For example, Table 2 gives values of s corresponding to various values of v_ϕ when ϕ is bounded between 0.175 and 0.70 rad (10°–40°).

Equation [11] can be generalized to yield the covariance between $\phi(z_i)$ and $\phi(z_j)$, for any two spatial points, z_i and z_j, as follows (Fenton and Griffiths 2008):

$$[13] \qquad \mathrm{Cov}[\phi(z_i),\ \phi(z_j)] \simeq (0.46)^2(\phi_{\max} - \phi_{\min})^2\frac{s^2\rho(z_i - z_j)}{4\pi^2 + s^2} = \sigma_\phi^2\rho(z_i - z_j)$$

where ρ is the correlation coefficient between the friction angle at a point z_i and a second point z_j. In this paper, a simple isotropic exponentially decaying correlation function will be assumed, having the form

$$[14] \qquad \rho(t) = \exp\left(-\frac{2|t|}{\theta}\right)$$

where $t = z_i - z_j$ is the distance between the two points. Note that the correlation function reflects the correlation between points in the underlying normally distributed random field, $G_\phi(z)$, and not directly between points in the friction field (although the correlation lengths in the different spaces are quite similar).

The next few sections of the paper will be making use of a variance reduction function, $\gamma(H)$, which specifies how the variance is reduced upon local averaging of ϕ over some length H. This function is defined to be the average correlation coefficient between every pair of points over the length H,

$$[15] \qquad \gamma(H) = \frac{1}{H^2}\int_0^H\int_0^H \rho(z_1 - z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$$

Fig. 1. Bounded distribution of friction angle normalized on the interval (0, 1).

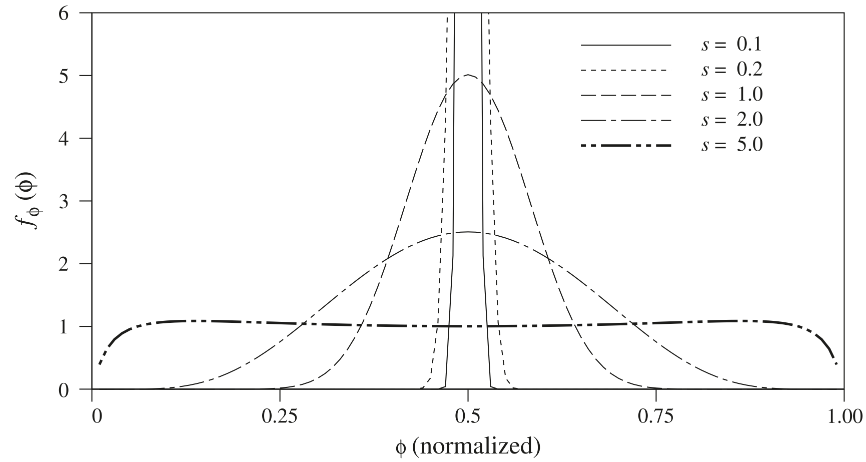


Table 2. Coefficients of variation of friction angle and corresponding s values for $\phi_{\min}$ = 0.175 rad (10°) and $\phi_{\max}$ = 70 rad (40°).

v_ϕ	s
0.1	1.16
0.2	2.44
0.3	4.07
0.344	5.00

Random load model

The load acting on a foundation is typically composed of dead loads, which are largely static, and live loads, which are largely dynamic. Dead loads are relatively well defined and can be computed by multiplying volumes by characteristic unit weights. The mean and variance of dead loads are thus reasonably well known. On the other hand, live loads are more difficult to characterize probabilistically. A typical definition of a live load is the maximum dynamic load (e.g., wind, vehicle, bookshelf loads, etc.) that a structure will experience during its design life. Note that the distribution of live load thus depends on the design lifetime. Dead and live loads will be denoted as F_D and F_L respectively. Assuming that the total load, F, is equal to the sum of the maximum lifetime live load, F_L, and the static dead load, F_D, i.e.,

$$[16] \qquad F = F_\mathrm{L} + F_\mathrm{D}$$

then the mean and variance of F, are given by

$$[17a] \qquad \mu_\mathrm{F} = \mu_\mathrm{L} + \mu_\mathrm{D}$$

$$[17b] \qquad \sigma_\mathrm{F}^2 = \sigma_\mathrm{L}^2 + \sigma_\mathrm{D}^2$$

where eq. [17b] is obtained under the reasonable assumption that the dead and live loads are independent.

The dead and live loads are assumed here to be lognormally distributed. The total load, $F = F_\mathrm{L} + F_\mathrm{D}$, is also assumed to be lognormally distributed, which was found to be a reasonable approximation by Fenton et al. (2008). The total load distribution thus has parameters,

$$[18a] \qquad \mu_{\ln F} = \ln(\mu_\mathrm{F}) - \frac{1}{2}\sigma_{\ln F}^2$$

$$[18b] \qquad \sigma_{\ln F}^2 = \ln\left(1 + \frac{\sigma_\mathrm{F}^2}{\mu_\mathrm{F}^2}\right)$$

The design problem considered in this study involves a pile supporting loads having means and standard deviations shown in Table 3. As will be shown in the section "Theoretical estimation of failure probability", the choice in mean values makes no difference to the resistance factors presented in this paper, so long as the coefficients of variation and the dead to live load ratio are maintained. The mean values shown in Table 3 are for illustrative purposed only. Reasonably conservative coefficients of variation of 30% for live loads and 15% for dead loads were assumed, and these are the important parameters in this study.

Assuming bias factors, k_D = 1.18 (Becker 1996) and k_L = 1.41 (Allen 1975), and importance factor, I_i = 1.0, gives the characteristic live load, $\hat{F}_\mathrm{L} = 1.41\mu_\mathrm{L}$ = 28.2 kN, dead load, $\hat{F}_\mathrm{D} = 1.18\mu_\mathrm{D}$ = 70.8 kN, and characteristic total factored design load, $\alpha_\mathrm{L}\hat{F}_\mathrm{L} + \alpha_\mathrm{D}\hat{F}_\mathrm{D} = 1.5\hat{F}_\mathrm{L} + 1.25\hat{F}_\mathrm{D}$ = (1.5)(28.2) + (1.25)(70.8) = 130.8 kN. These numbers are for illustration only, since the mean values can be scaled without changing the results of the paper.

Theoretical estimation of failure probability

To estimate the probability of failure of a pile, the soil is first modeled as a spatially varying random field. This study considers a two-dimensional random field in which the pile is placed vertically at a certain position and soil samples, as in a CPT or SPT sounding, are taken vertically at some, possibly different, position. The theory required to estimate the failure probability of a pile in soils under effective stress conditions can be explained as follows. When the soil properties are spatially variable, as they are in reality, then eq. [6] can be replaced by

$$[19] \qquad R_\mathrm{u} = \frac{1}{2}pa\gamma H^2(1 - \sin\overline{\phi})\tan(b\overline{\phi})$$

where $\overline{\phi}$ is the equivalent friction angle of the soil, defined as the uniform (constant) friction angle, which leads to the same resistance as observed in the spatially varying soil over the pile

Table 3. Load distribution parameters.

μ_L (kN)	μ_D (kN)	σ_L (kN)	σ_D (kN)	μ_F (kN)	σ_F (kN)	$\mu_{\ln F}$	$\sigma_{\ln F}$
20	60	6	9	80	10.82	4.4	0.14

length, H. It is assumed here that $\overline{\phi}$ is the arithmetic average of the spatially variable friction angle over the pile length, H,

$$[20] \qquad \overline{\phi} = \frac{1}{H}\int_0^H \phi(z)\,\mathrm{d}z \simeq \frac{1}{n}\sum_{i=1}^{n}\overline{\phi}_i$$

where $\phi(z)$ is interpreted as the average friction angle of the soil around the pile perimeter at depth z. If the pile is broken up into a series of elements (as will be done in the simulation), $\overline{\phi}$ is determined using the sum at the right of eq. [20], in which $\overline{\phi}_i$ is the local average of $\phi(z)$ over the ith element, for $i = 1, \ldots, n$.

The required minimum design pile length, H, can be obtained by substituting eq. [9] into eq. [7] (taking $I_i = 1.0$ and considering just dead and live loads),

$$[21] \qquad \varphi_{gu}\left[\frac{1}{2}pa\gamma H^2(1-\sin\hat{\phi})\tan(b\hat{\phi})\right] = \alpha_L\hat{F}_L + \alpha_D\hat{F}_D$$

which leads to

$$[22] \qquad H = \sqrt{\frac{2(\alpha_L\hat{F}_L + \alpha_D\hat{F}_D)}{\varphi_{gu}pa\gamma(1-\sin\hat{\phi})\tan(b\hat{\phi})}}$$

By substituting eq. [22] into eq. [19], the ultimate geotechnical resistance, R_u, can be written as

$$[23] \qquad R_u = \left(\frac{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D}{\varphi_{gu}}\right)\frac{(1-\sin\overline{\phi})[\tan(b\overline{\phi})]}{(1-\sin\hat{\phi})[\tan(b\hat{\phi})]}$$

The reliability-based design goal in this study is to find the required length H such that the probability that the actual load, F, exceeds the actual resistance, R_u, is less than some small acceptable failure probability, p_m. The actual failure probability, p_f, is

$$[24] \qquad p_f = \mathrm{P}[F > R_u]$$

and a successful design methodology will have $p_f \le p_m$. Substituting eq. [23] into eq. [24] leads to

$$[25] \qquad p_f = \mathrm{P}\left[F > \frac{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D}{\varphi_{gu}}\left[\frac{(1-\sin\overline{\phi})\tan(b\overline{\phi})}{(1-\sin\hat{\phi})\tan(b\hat{\phi})}\right]\right] = \mathrm{P}\left[\frac{F(1-\sin\hat{\phi})\tan(b\hat{\phi})}{(1-\sin\phi)\tan(b\phi)} > \frac{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D}{\varphi_{gu}}\right]$$

where in the last step, all remaining random quantities were moved to the left-hand side (LHS) of the inequality.

Note that the parameters a, γ, and p have all cancelled out of the above failure probability estimate. This means that the values of these parameters will not affect the required resistance factors obtained in this study. It is also instructive to investigate how varying the mean load and mean friction angle might influence the failure probability, and hence the required resistance factor. If eq. [25] is rearranged so that the load terms are grouped together, and similarly for the resistance terms, one gets

$$[26] \qquad p_f = \mathrm{P}\left[\frac{F}{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D} > \frac{1}{\varphi_{gu}}\left[\frac{(1-\sin\overline{\phi})\tan(b\overline{\phi})}{(1-\sin\hat{\phi})\tan(b\hat{\phi})}\right]\right]$$

which involves comparing the distribution of the load term on the LHS of the inequality to the distribution of the friction angle term on the RHS. Interest is in how these two quantities are affected by changes in their means. The mean of the load term is (see eq. [17a] and discussion at the end of the section "Random load model"),

$$[27] \qquad \mathrm{E}\left[\frac{F}{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D}\right] = \frac{\mu_L + \mu_D}{\alpha_L k_L\mu_L + \alpha_D k_D\mu_D} = \frac{1 + R_{D/L}}{\alpha_L k_L + \alpha_D k_D R_{D/L}}$$

from which it can be seen that the mean of the load term depends only on the ratio of the dead to live load means, $R_{D/L} = \mu_D/\mu_L$, and not on the actual load means. Thus, the load means can be scaled by any common amount without affecting the overall pile failure probability predicted here.

The RHS of the inequality in eq. [26] involves the random quantity $(1-\sin\overline{\phi})\tan(b\overline{\phi})/(1-\sin\hat{\phi})\tan(b\hat{\phi})$, which to first order has mean

$$[28] \qquad \frac{(1-\sin\mu_\phi)\tan(b\mu_\phi)}{(1-\sin\mu_\phi)\tan(b\mu_\phi)} = 1$$

which is independent of the choice in the mean friction angle, μ_ϕ, and the choice in the parameter b. In other words, these parameters can be changed without significantly affecting the results of this study (at least to first order).

Attention is now turned to obtaining a solution to the failure probability given by eq. [25]. To simplify the notation, the following variables corresponding to the elements of eq. [25] will be defined,

$$[29a] \qquad \hat{X} = (1-\sin\hat{\phi})\tan(b\hat{\phi})$$

$$[29b] \qquad \overline{X} = (1-\sin\overline{\phi})\tan(b\overline{\phi})$$

$$[29c] \qquad q = \alpha_L\hat{F}_L + \alpha_D\hat{F}_D$$

$$[29d] \qquad Y = \frac{F\hat{X}}{\overline{X}}$$

which means that eq. [25] can be reexpressed as

$$[30] \qquad p_f = \mathrm{P}[Y > q/\varphi_{gu}]$$

To compute the probability in eq. [30], the form of the distribution of Y must first be determined (e.g., lognormal, normal, etc.), followed by the specification of its parameters. Since F is the sum of two lognormally distributed random variables (F_L and F_D), and both $\hat{X}$ and $\overline{X}$ are nonlinear transformations of arithmetic averages, the exact distribution of Y cannot easily be derived analytically. The central limit theorem suggests a lognormal distribution since eq. [29d] is a product of positive random variables. Figure 2, based on a simulation of 100 000 realizations of F, $\hat{X}$ and $\overline{X}$, presents a histogram of Y along with a fitted lognormal distribution. The simulation was performed by generating two-dimensional random friction angle fields, using the parameters given in Table 2 with a "worst-case" correlation length $\theta = 6$ m (the worst-case correlation length will be discussed in the section "Geotechnical resistance factors"), and realizations of the load F using parameters given in Table 3. Clearly, the lognormal hypothesis is very reasonable and will be adopted here.

Since Y is well approximated by a lognormal distribution, its logarithm

[31] $$\ln Y = \ln F + \ln\hat{X} - \ln\overline{X}$$

is approximately normally distributed and p_f can be found from

[32] $$p_\mathrm{f} = \mathrm{P}[(Y > q/\varphi_\mathrm{gu}] = \mathrm{P}[\ln Y > \ln(q/\varphi_\mathrm{gu})] = 1 - \Phi\left[\frac{\ln(q/\varphi_\mathrm{gu}) - \mu_{\ln Y}}{\sigma_{\ln Y}}\right]$$

where Φ is the standard normal cumulative distribution function.

The failure probability p_f in eq. [32] can be estimated once the mean and variance of ln Y are determined. These are given by

[33*a*] $$\mu_{\ln Y} = \mu_{\ln F} + \mu_{\ln\hat{X}} - \mu_{\ln\overline{X}}$$

[33*b*] $$\sigma^2_{\ln Y} = \sigma^2_{\ln F} + \sigma^2_{\ln\hat{X}} + \sigma^2_{\ln\overline{X}} - 2\mathrm{Cov}(\ln\overline{X}, \ln\hat{X})$$

where the total load, F, and friction angle, ϕ, are assumed to be independent. The components of eq. [33] can be computed as follows:

1. Assuming that the total load F is equal to the sum of the maximum live load, F_L, acting over the lifetime of the structure and the static dead load, F_D, i.e., $F = F_\mathrm{L} + F_\mathrm{D}$, both of which are random, then

[34*a*] $$\mu_{\ln F} = \ln(\mu_\mathrm{F}) - \frac{1}{2}\sigma^2_{\ln F}$$

[34*b*] $$\sigma^2_{\ln F} = \ln\left(1 + \frac{\sigma^2_\mathrm{F}}{\mu^2_\mathrm{F}}\right)$$

where $\mu_\mathrm{F} = \mu_\mathrm{L} + \mu_\mathrm{D}$ is the sum of the mean live and dead loads, and σ^2_F is the variance of the total load defined by

[35] $$\sigma^2_\mathrm{F} = \sigma^2_\mathrm{L} + \sigma^2_\mathrm{D}$$

assuming dead and live loads to be independent.

2. With reference to eq. [8] and the fact that the friction angle random field is assumed to be stationary,

[36] $$\mu_{\hat{\phi}} = \mathrm{E}\left[\frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_i\right] = \frac{1}{m}\sum_{i=1}^{m}\mu_\phi = \mu_\phi$$

where E is the expectation operator. The mean and variance of $\ln\hat{X}$ can be obtained by using eq. [36] and a third-order Taylor series approximation to eq. [29*a*] as follows (Naghibi 2010):

[37*a*] $$\mu_{\ln\hat{X}} \simeq \ln[(1 - \sin\mu_\phi)\tan(b\mu_\phi)] + \frac{\sigma^2_{\hat{\phi}} d_2}{2}$$

[37*b*] $$\sigma^2_{\ln\hat{X}} \simeq d_1^2\sigma^2_{\hat{\phi}} + \left(\frac{d_2^2}{2} + d_1 d_3\right)\sigma^4_{\hat{\phi}} + \frac{5d_3^2\sigma^6_{\hat{\phi}}}{12}$$

where d_1, d_2, and d_3 are derivatives of $\ln\hat{X}$ given by eq. [A1]. The variance of $\hat{\phi}$ can be obtained from

[38] $$\sigma^2_{\hat{\phi}} \simeq \frac{\sigma^2_\phi}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\rho(z_i^\mathrm{o} - z_j^\mathrm{o})$$

where σ_ϕ is given by eq. [11], z_i^o is the spatial location of the center of the ith soil sample ($i = 1, 2, \ldots, m$), and ρ is the correlation function defined by eq. [14]. The approximation in eq. [38] arises because correlation coefficients between the local averages associated with observations are approximated by correlation coefficients between the local average centers (Fig. 3). Assuming that $\hat{\phi}$ actually represents a local average of ϕ over a sample length of size $D = \Delta z \times m$, where D is the depth over which the samples are taken, m is the number of observations over sample depth D, and Δz is the vertical dimension of each observation, then $\sigma^2_{\hat{\phi}}$ is probably more accurately computed as

[39] $$\sigma^2_{\hat{\phi}} = \sigma^2_\phi\gamma(D)$$

where $\gamma(D)$ is the variance reduction function that measures the reduction in variance due to local averaging over the sample length D, as given by eq. [15]. All angles are measured in radians, including those used in eqs. [11] and [13].

3. With reference to eq. [20],

[40] $$\mu_{\overline{\phi}} = \mathrm{E}\left[\frac{1}{H}\int_0^H \phi(z)\,\mathrm{d}z\right] = \frac{1}{H}\int_0^H \mu_\phi\,\mathrm{d}z = \mu_\phi$$

By considering eqs. [40] and [29*b*], the mean and variance of $\ln\overline{X}$ can be obtained in the same fashion as for $\ln\hat{X}$ (in fact, they only differ due to differing local averaging in the variance calculation),

[41*a*] $$\mu_{\ln\overline{X}} \simeq \ln[(1 - \sin\mu_\phi)\tan(b\mu_\phi)] + \frac{\sigma^2_{\overline{\phi}} d_2}{2}$$

[41*b*] $$\sigma^2_{\ln\overline{X}} \simeq d_1^2\sigma^2_{\overline{\phi}} + \left(\frac{d_2^2}{2} + d_1 d_3\right)\sigma^4_{\overline{\phi}} + \frac{5d_3^2\sigma^6_{\overline{\phi}}}{12}$$

[41*c*] $$\sigma^2_{\overline{\phi}} \simeq \sigma^2_\phi\gamma(H)$$

where d_1, d_2, d_3, and $\gamma(H)$ are defined by eqs. [A1*a*], [A1*b*], [A1*c*], and [15], respectively.

4. The covariance between $\ln\hat{X}$ over the sample depth, $D = \Delta z \times m$, and $\ln\overline{X}$ along the pile length, $H = \Delta z \times n$, in eq. [33*b*], is approximated by

[42] $$\mathrm{Cov}(\ln\overline{X}, \ln\hat{X}) \simeq d_1^2\sigma^2_\phi\gamma_{HD} + \sigma^2_\phi\gamma_{HD}\left[\frac{d_1 d_3}{2}(\sigma^2_{\hat{\phi}} + \sigma^2_{\overline{\phi}}) + \frac{d_3^2}{4}\sigma^2_{\hat{\phi}}\sigma^2_{\overline{\phi}}\right] + \frac{d_2^2}{2}(\sigma^2_\phi\gamma_{HD})^2 + \frac{d_3^2}{6}(\sigma^2_\phi\gamma_{HD})^3$$

Fig. 2. Frequency histogram of Y (solid line) and fitted lognormal distribution (dotted line).

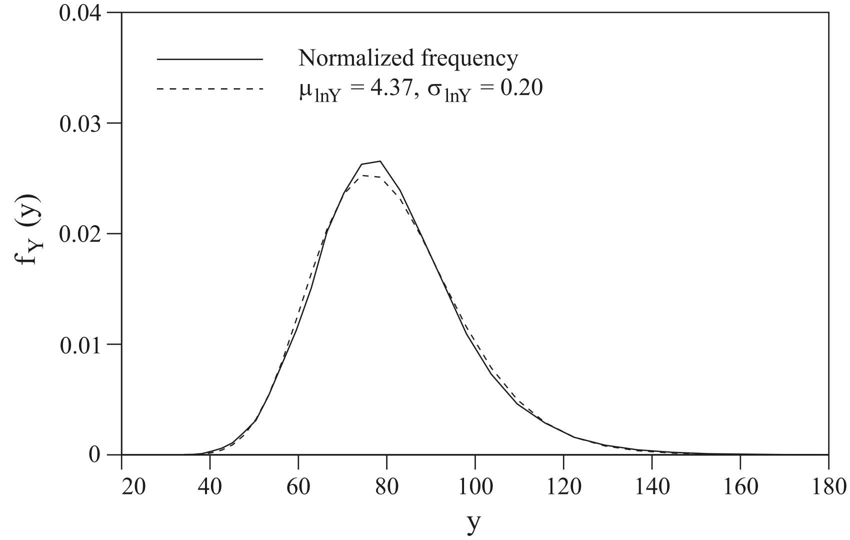
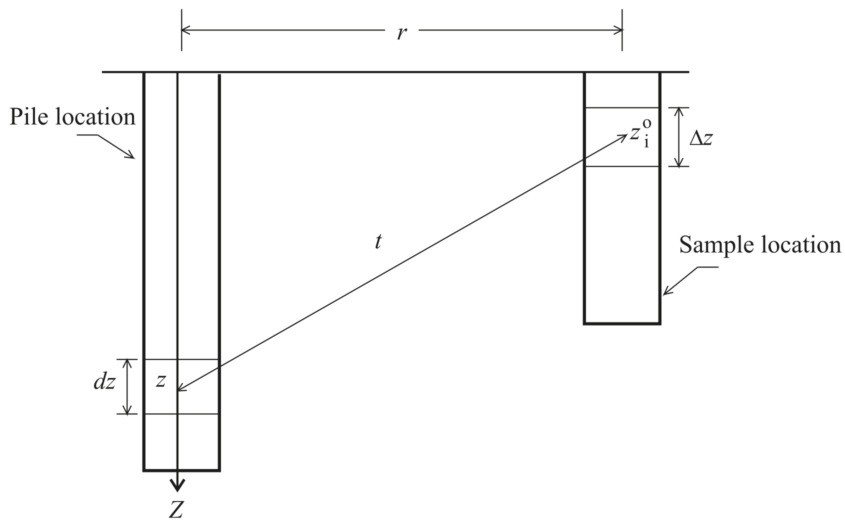


Fig. 3. Correlation between local averages is approximated by the correlation function, $\rho(t)$, between centers.



where γ_{HD} is the average correlation coefficient between the soil in the sample length, D, and the soil along the pile length, H. Assuming that the pile is broken up into a series of n elements, γ_{HD} is given by

$$[43] \qquad \gamma_{HD} \simeq \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\rho\left[\sqrt{r^2+(z_j-z_i^{\mathrm{o}})^2}\,\right]\mathrm{d}z$$

where r is the horizontal distance between the pile centerline and the centerline of the soil sample column, z_j is the depth in which the center of j and z_i^{o} is the spatial location of the center of the ith soil sample (see Fig. 4). The approximation in the covariance (eq. [42]) arises both because of the use of a third-order Taylor series approximation and because correlation coefficients between local averages associated with observations are approximated by correlation coefficients between the local average centers.

Substituting eqs. [34], [37], [41], and [42] into eq. [33] leads to

$$[44a] \qquad \mu_{\ln Y} \simeq \mu_{\ln F} + \frac{d_2}{2}(\sigma^2_{\hat{\phi}} - \sigma^2_{\bar{\phi}})$$

$$[44b] \qquad \sigma^2_{\ln Y} \simeq \sigma^2_{\ln F} + d_1^2(\sigma^2_{\bar{\phi}} + \sigma^2_{\hat{\phi}}) + \left(\frac{d_2^2}{2} + d_1 d_3\right)(\sigma^4_{\bar{\phi}} + \sigma^4_{\hat{\phi}}) + \frac{5d_3^2}{12}(\sigma^6_{\bar{\phi}} + \sigma^6_{\hat{\phi}}) - 2\mathrm{Cov}(\ln\overline{X},\ \ln\hat{X})$$

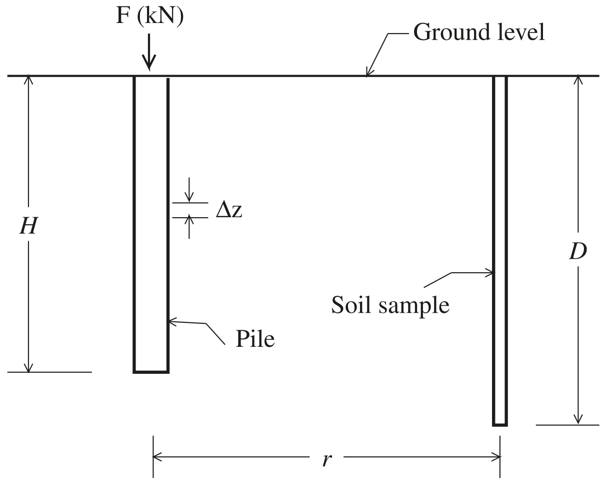
The argument to Φ in eq. [32] is the reliability index,

$$[45] \qquad \beta = \frac{\ln(q/\varphi_{\mathrm{gu}}) - \mu_{\ln Y}}{\sigma_{\ln Y}}$$

If the reliability index is specified through knowledge of p_{m}, then the geotechnical resistance factor is determined by

$$[46] \qquad \varphi_{\mathrm{gu}} = \exp(\ln q - \mu_{\ln Y} - \beta\sigma_{\ln Y})$$

Fig. 4. Relative locations of pile and soil samples.



Comparison of theoretical and simulated failure probabilities

To test the proposed theory, a series of $n_{sim} = 10\,000$ realizations of a soil mass are simulated for each of a range of soil variability parameters and sampling distances, r. The resulting Monte Carlo simulation-based failure probability estimates are then compared to the theory presented in the section "Theoretical estimation of failure probability".

In detail the Monte Carlo simulation proceeds as follows:

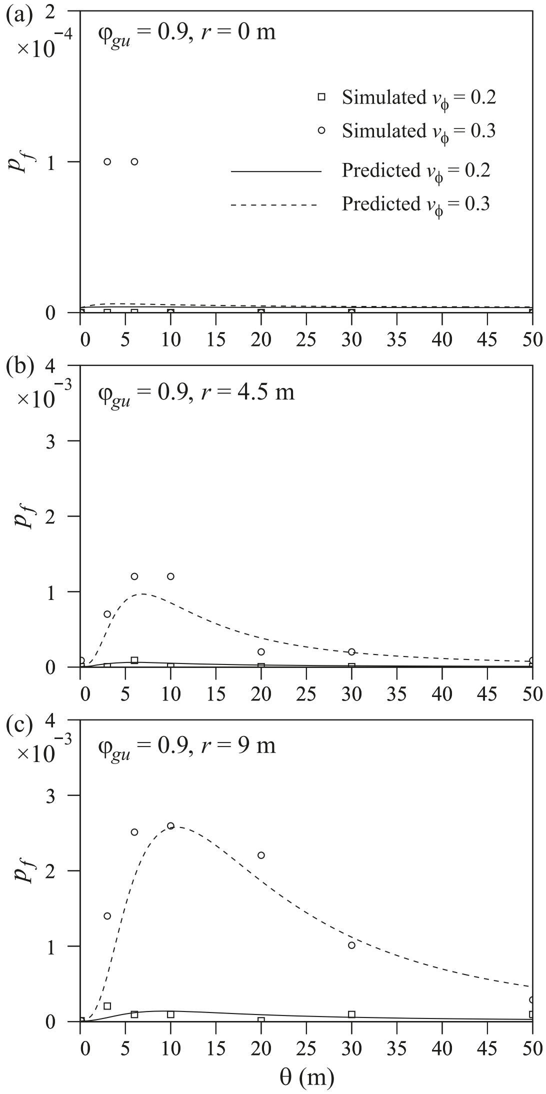
1. The friction angle, ϕ, of a soil mass is simulated as a spatially variable random field using the local average subdivision (LAS) method (Fenton and Vanmarcke 1990). The number of soil cells in the X and Y directions are assumed to be 128×128, and each cell size dimension is taken to be 0.1×0.1. The correlation length is varied from 0 to 50 m, and two coefficients of variation of friction angle, v_ϕ, are considered: $v_\phi = 0.2$ ($s = 2.44$) and $v_\phi = 0.3$ ($s = 4.07$). The friction angle is assumed to have a bounded distribution from $\phi_{min} = 0.175$ rad (10°) to $\phi_{max} = 0.70$ rad (40°).
2. The simulated soil is sampled along a vertical line through the soil at some distance, r, from the pile. These virtually sampled soil properties are used to estimate the characteristic friction angle, $\hat{\phi}$, according to eq. [8]. Three sampling distances are considered as illustrated in Fig. 4: the first is at $r = 0$ m, which means that the samples are taken at the pile location. In this case, uncertainty about the pile resistance only arises if the pile extends below the sampling depth. Typically, probabilities of failure when $r = 0$ m are very small. The other two sample distances considered are $r = 4.5$ and 9.0 m, corresponding to reduced understanding of the soil conditions at the pile location (see Fig. 4). These rather arbitrary distances were based on preliminary random field simulations, which happened to involve fields 9 m in width. However, it is really the ratio, r/θ, that governs the failure probability. No attempt is made here to include the effects of measurement error nor of errors in mapping actual observations, e.g., CPT values, to engineering properties such as friction angle. Thus, the predicted failure probability (either from theory or simulation) will be somewhat unconservative (failure probability increases as measurement error increases). However, both the theoretical technique and the simulation treat measurement errors in the same way, allowing a consistent comparison between the two. In addition, since increasing r results in an increased conditional variance at the pile location, r can be used as a proxy to represent measurement and other sources of error — simply set r to a value larger than the actual distance. Similarly, using a higher coefficient of variation in the friction angle distribution will produce a similar result.
3. The required design pile length, H, is calculated using eq. [22].
4. Dead and live loads, F_D and F_L, respectively, are simulated as independent lognormally distributed random variables and then added to produce the actual total load on the pile, $F = F_L + F_D$. The means and standard deviations of the dead and live loads are assumed to be as given in Table 3. As shown above, the mean loads do not affect the failure probabilities — it is only the coefficients of variation (e.g., σ/μ) that are important.
5. The true ultimate pile resistance, R_u, is computed using eq. [6] by considering the simulated soil properties along the pile.
6. The ultimate resistance, R_u, and total load F are compared. If $F > R_u$, then the pile, as designed, is assumed to have failed.
7. The entire process from steps 1–6 is repeated n_{sim} times ($n_{sim} = 10\,000$ in the present study). If n_f of these repetitions result in a pile failure, then an estimate of the probability of failure is $p_f \simeq n_f/n_{sim}$.
8. Repeating steps 1–7 using various values of φ_{gu} in the design step allows plots of failure probability versus geotechnical resistance factor to be produced for the various sampling distances, coefficients of variation of the friction angle, and correlation length.

The comparison between the probabilistic analyses of piles using Monte Carlo simulation based on 10 000 realizations with those computed theoretically by eq. [32] are illustrated in Fig. 5. The parameters $a = 1.2$, $b = 0.8$ are used in the simulation, but as mentioned above, they effectively cancel out of the failure probability calculation so that the results in Fig. 5 are quite general.

It is immediately clear from Fig. 5 that the probability of failure, p_f, increases with soil variability, v_ϕ, which is to be expected. Also, as expected, the probabilities of failure are smaller when the soil is sampled directly at the pile than when sampled some distance away from the pile centerline. This means that considerable construction savings can be achieved by improving the sampling scheme, especially when significant soil variability exists.

The worst agreement between theory and simulation occurs when the soil is sampled at the pile location (Fig. 5a, $r = 0$ m). This may be largely due to estimator error in the simulations. For example, for those simulations having one failure out of 10 000, the estimated probability of failure is $p_f = 10^{-4}$, which has standard error, $\sigma_{\hat{p}_f} = \sqrt{(10^{-4})(0.9999)/10\,000} \simeq 10^{-4}$. This means that the simulation cannot be used to validate small probabilities, i.e., probabilities about 10^{-4} or less — the simulation-based probability estimates become highly uncertain. The potential for large estimator error is seen in Fig. 5a, where most failure probability estimates are zero,

Fig. 5. Comparison of failure probabilities estimated by simulation (10 000 realizations) and theory for geotechnical resistance factor (φ_{gu} = 0.9) and three sampling locations: (*a*) r = 0 m; (*b*) r = 4.5 m; (*c*) r = 9 m.



except for those worst cases where one out of the 10 000 realizations failed.

It can be seen from Fig. 5 that the agreement between theory and simulation is reasonable and that the larger discrepancies can easily be attributed to estimator error. The overall agreement implies that the theory can be used to estimate pile failure probabilities. The theory will thus be used in the following section to provide recommendations regarding geotechnical resistance factors required to achieve certain target maximum acceptable lifetime failure probabilities.

Geotechnical resistance factors

In this section, the geotechnical resistance factor, φ_{gu}, required to achieve four maximum acceptable failure probability levels, (10^{-2}, 10^{-3}, 10^{-4}, and 10^{-5}), will be investigated. The corresponding reliability indices of these four target probabilities are approximately 2.3, 3.1, 3.7, and 4.3, respectively.

Figures 6–8 show the geotechnical resistance factors required for the cases where the soil is sampled at the pile location r = 0, at a distance of r = 4.5 m and at a distance of r = 9 m from the pile centerline. Four coefficients of variation, $v_\phi = 0.1$ (s = 1.16), $v_\phi = 0.2$ (s = 2.44), $v_\phi = 0.3$ (s = 4.07), and $v_\phi = 0.344$ (s = 5) are considered for each of the three sampling locations.

In the cases where the samples are taken at the pile location so the soil conditions are well understood, the geotechnical resistance factor exceeds 1.0 when $p_m \geq 10^{-3}$. In the cases where the samples are taken 4.5 and 9 m from pile centerline, the geotechnical resistance factor exceeds 1.0 when $p_m \geq 10^{-2}$. The cases where $\varphi_{gu} > 1.0$ are not shown.

The worst case values of geotechnical resistance factors occur when the correlation length, θ, is between about 1 and 10 m. Since the correlation length is very hard to estimate (requiring large volumes of data), the occurrence of a worst case means that failure probabilities, and required resistance factors, can be conservatively estimated using the worst-case correlation length — without having to know the actual correlation length at the site.

As seen in Fig. 8, the smallest geotechnical resistance factors correspond to the smallest acceptable failure probability considered, $p_m = 10^{-5}$, when the soil is sampled 9 m away from the pile centerline, as expected. When the friction angle coefficient of variation, v_ϕ, is relatively large ($v_\phi = 0.344$) the worst-case values of φ_{gu} dip down to 0.57 to achieve $p_m = 10^{-5}$. In other words, there will be a significant construction cost penalty if a highly reliable pile is to be designed using a site investigation that is insufficient to reduce the residual variability to less than $v_\phi = 0.344$ and which is taken at a fairly large distance from the pile (e.g., r = 9 m).

The worst-case values of geotechnical resistance factors required to achieve the indicated maximum acceptable failure probabilities, as seen in Figs. 6–8, are summarized in Table 4. Some of the geotechnical resistance factors recommended in this study for $p_m = 10^{-2}$ and $p_m = 10^{-3}$ are greater than 1.0, which may be because the load factors provide too much safety for the larger acceptable failure probabilities when the site is well understood.

To maintain a constant level of reliability in eq. [7], changes in the load factors must be accompanied by a corresponding changes in the required geotechnical resistance factor, φ_{gu}. In other words, if one is to compare the geotechnical resistance factors, φ_{gu}, recommended here with values provided in other code documents, the ratio of the total load factor to the geotechnical resistance factor, α_T/φ_{gu}, which is an estimate of the overall factor of safety used by each document, must be considered. The total load factor, α_T, is defined as

$$\alpha_T = \frac{\alpha_L \hat{F}_L + \alpha_D \hat{F}_D}{\hat{F}_L + \hat{F}_D} \tag{47}$$

where α_L and α_D are the live and dead load factors, respectively, and $\hat{F}_L$ and $\hat{F}_D$ are the characteristic live and dead loads, respectively.

The dead load factor, $\alpha_D = 1.25$, and live load factor, $\alpha_L = 1.5$, used in this paper, are as specified by the National Building Code of Canada (NBCC) (National Research Council Canada 2005). Bias factors of $k_D = 1.18$ (Becker 1996),

Fig. 6. Geotechnical resistance factors when the soil has been sampled at the pile location (r = 0 m) (note the reduced vertical scale): (*a*) $p_m = 10^{-4}$; (*b*) $p_m = 10^{-5}$.

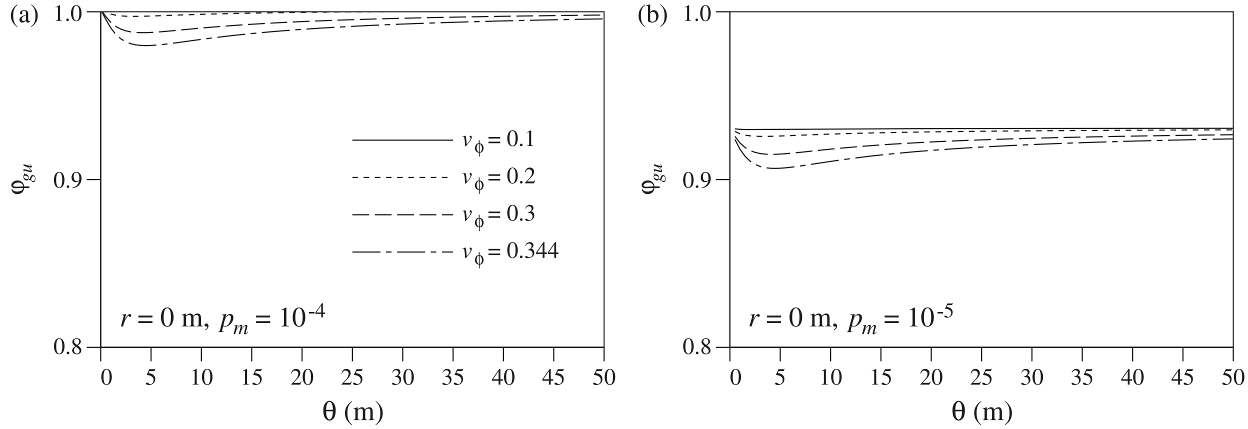


Fig. 7. Geotechnical resistance factors when the soil has been sampled (r = 4.5 m) from the pile centerline: (*a*) $p_m = 10^{-3}$; (*b*) $p_m = 10^{-4}$; (*c*) $p_m = 10^{-5}$.

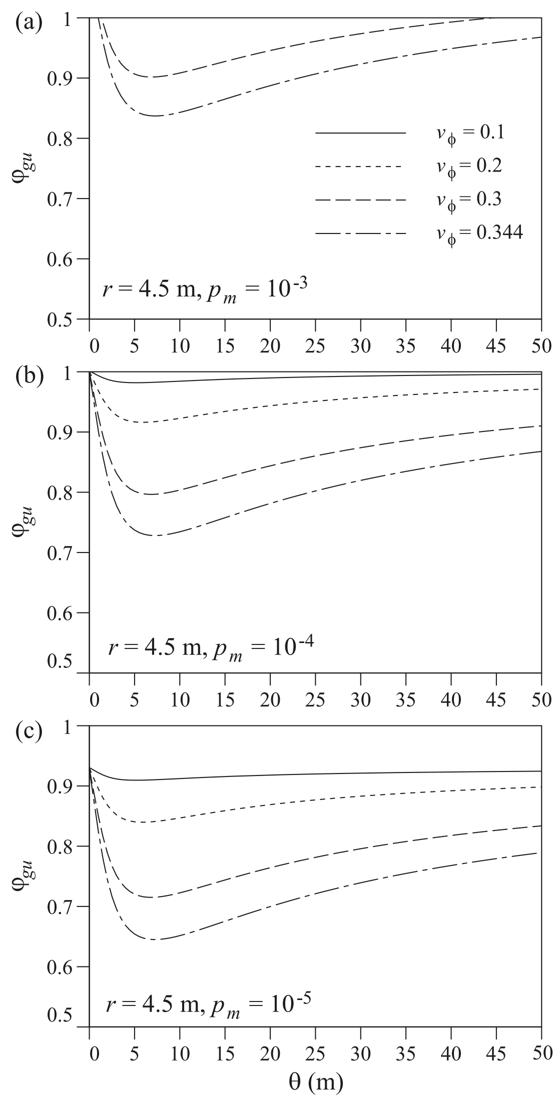


Fig. 8. Geotechnical resistance factors when the soil has been sampled (r = 9 m) from the pile centerline: (*a*) $p_m = 10^{-3}$; (*b*) $p_m = 10^{-4}$; (*c*) $p_m = 10^{-5}$.

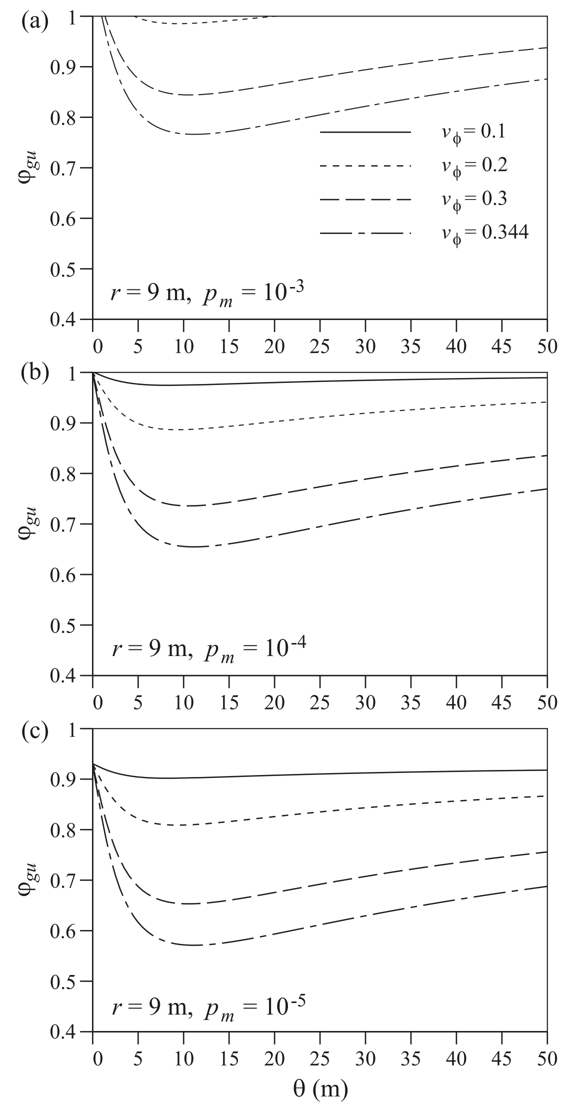


Table 4. Worst-case geotechnical resistance factors for various coefficients of variation, v_ϕ, distance to sampling location, r, and acceptable failure probabilities, p_m.

		Geotechnical resistance factor			
r (m)	v_ϕ	$p_m = 10^{-2}$	$p_m = 10^{-3}$	$p_m = 10^{-4}$	$p_m = 10^{-5}$
0.0	0.1	1.21	1.09	1.00	0.93
0.0	0.2	1.20	1.08	0.99	0.93
0.0	0.3	1.19	1.06	0.99	0.92
0.0	0.344	1.17	1.04	0.98	0.91
4.5	0.1	1.19	1.08	0.98	0.91
4.5	0.2	1.15	1.02	0.92	0.84
4.5	0.3	1.06	0.90	0.80	0.71
4.5	0.344	1.00	0.85	0.72	0.64
9.0	0.1	1.19	1.07	0.98	0.90
9.0	0.2	1.13	0.99	0.89	0.81
9.0	0.3	1.02	0.85	0.74	0.65
9.0	0.344	0.93	0.77	0.65	0.57

$k_L = 1.41$ (Allen 1975), and the ratio of dead to live load means, $R_{D/L} = 3.0$, are assumed here. The characteristic dead to live load ratio, $\hat{R}_{D/L}$, and the total load factor, α_T, in this paper are then

$$[48a] \qquad \hat{R}_{D/L} = \frac{\hat{F}_D}{\hat{F}_D} = \frac{k_D\mu_D}{k_L\mu_L} = \frac{1.18(3\mu_L)}{1.41\mu_L} = \frac{1.18(3)}{1.41} = 2.5$$

$$[48b] \qquad \alpha_T = \frac{\alpha_L\hat{F}_L + \alpha_D\hat{F}_D}{\hat{F}_L + \hat{F}_D} = \frac{\alpha_L + \alpha_D\hat{R}_{D/L}}{1 + \hat{R}_{D/L}} = \frac{1.5 + 1.25(2.5)}{1 + 2.5} = 1.32$$

Table 5 compares the total load factor to resistance factor ratios recommended in this study with those recommended by the Canadian Foundation Engineering Manual (CFEM) (Canadian Geotechnical Society 1992), NBCC (National Research Council Canada 2005), now also referenced by the 2006 CFEM (Canadian Geotechnical Society 2006), the Canadian Highway Bridge Design Code (CHBDC) (Canadian Standards Association 2006), the Australian Standard Bridge Design Part 3 (AS 5100.3, Standards Australia 2004), two versions of the American Association of State Highway and Transportation Officials codes (AASHTO 2002, 2007), and the National Cooperative Highway Research Program (NCHRP 507, Paikowsky 2004). The geotechnical resistance factors recommended in the current study occupy the first six rows of Table 5 and correspond to the cases where $v_\phi = 0.344$, and samples are taken 4.5 and 9 m from the pile centerline for maximum acceptable failure probabilities, $p_m = 10^{-3}$, 10^{-4}, and 10^{-5}.

The total load factor to resistance factor ratio, α_T/φ_{gu}, is generally higher in other documents than predicted here, although when $r = 9$ m and $p_m = 10^{-5}$, the 2.32 given by this study is comparable to the 1992 CFEM (Canadian Geotechnical Society 1992) (2.62) and the lower end of the Australian standard AS 5100.3 (Standards Australia 2004) (2.45). With respect to the Canadian documents (CFEM, NBCC, and CHBDC), the higher load to resistance factor ratios could be due to some extent to the fact that the resistance factors provided in these documents do not distinguish between frictional and cohesive soils, and presumably were selected conservatively to provide sufficient reliability for any soil type.

On the other hand, the Australian standard, AASHTO, and NCHRP documents do distinguish between cohesive and frictional soils, and their α_T/φ_{gu} ratios are significantly higher (more conservative) than suggested by this research. The most likely reason for this discrepancy is that this study neglects the effect of measurement error. For frictional soils, measurement error can be significant, due to sample disturbance and (or) correlation errors, and the friction angle used in design is often quite a bit more uncertain than assumed here. One way of accounting for measurement error in this research is to calibrate the results against current standards. In other words, the higher conservatism in existing codes implies that the first six rows of Table 5 must be viewed in a relative sense. For example, if it is assumed that the 2005 NBCC (National Research Council Canada 2005) and 2006 CHBDC (Canadian Standards Association 2006) codes are aimed at a site understanding equivalent to the $r = 9$ m used here and to a maximum acceptable failure probability of about $p_m = 10^{-4}$ (which are both deemed to be reasonable assumptions), then the results in the first six rows of Table 5 should be scaled so that they approximately match the NBCC and CHBDC result of about $\alpha_T/\varphi_{gu} = 3.3$ when $r = 9$ m and $p_m = 10^{-4}$. The actual scaling factor is the ratio of the resistance factor required to achieve $\alpha_T/\varphi_{gu} \simeq 3.3$ at $r = 9$ m and $p_m = 10^{-4}$ to the corresponding value in Table 5 recommended by the more recent Canadian codes. Based on Table 5, the required ratio is $0.4/0.65 \simeq 0.615$, which when multiplying the resistance factors listed in Table 4, yield a set of resistance factors that are properly calibrated to existing practice. The resulting calibrated factors listed in Table 6 now provide a quantitative estimate of how the resistance factors should change as the degree of site understanding (measured by r) and failure consequence (measured by p_m) change. The factors in Table 6 can thus be used to determine how code factors should be changed from currently acceptable values as site understanding and target reliability changes.

Table 5. Comparison of geotechnical resistance factors determined in this study (first six rows) to those recommended by other sources.

Source	$\hat{R}_{D/L}$	α_L	α_D	α_T	φ_{gu}	α_T/φ_{gu}
r = 4.5 m, $p_m = 10^{-3}$	2.5	1.50	1.25	1.32	0.85	1.55
r = 4.5 m, $p_m = 10^{-4}$	2.5	1.50	1.25	1.32	0.72	1.83
r = 4.5 m, $p_m = 10^{-5}$	2.5	1.50	1.25	1.32	0.64	2.06
r = 9.0 m, $p_m = 10^{-3}$	2.5	1.50	1.25	1.32	0.77	1.71
r = 9.0 m, $p_m = 10^{-4}$	2.5	1.50	1.25	1.32	0.65	2.03
r = 9.0 m, $p_m = 10^{-5}$	2.5	1.50	1.25	1.32	0.57	2.32
CFEM (1992)[a]	3.0	1.50	1.25	1.31	0.50	2.62
NBCC (2005)[b]	3.0	1.50	1.25	1.31	0.40	3.28
CHBDC (2006)[c]	3.0	1.70	1.20	1.33	0.40	3.33
AS 5100.3 (2004)[d]	3.0	1.80	1.20	1.35	0.40–0.55	2.45–3.38
AASHTO (2002)	3.7	2.86	1.30	1.63	0.50	3.26
AASHTO (2007)	3.7	1.75	1.25	1.36	0.25	5.44
NCHRP 507 (2004)[e]	2.0	1.75	1.25	1.42	0.25–0.40	3.55–5.68

[a]Canadian Geotechnical Society (1992).
[b]National Research Council Canada (2005).
[c]Canadian Standards Association (2006).
[d]Standards Australia (2004).
[e]Paikowsky (2004).

Conclusions

The paper presents the theory required to estimate deep foundation failure probability as a function of ground variability, v_ϕ, site understanding (as reflected by sampling distance), and target lifetime failure probability. The theory is validated by random soil and load simulation and can be used to complement code calibration efforts aimed at determining required geotechnical resistance factors for the ultimate limit state design of deep foundations under effective stress (frictional) conditions.

The overall goal of any LRFD is to achieve a certain target maximum acceptable failure probability that depends on the severity of failure consequences. The resistance factor thus depends both on how well the ground conditions are understood at the pile location and on the target failure probability. A fairly wide range, from 10^{-2} to 10^{-5}, in maximum acceptable lifetime failure probabilities were investigated. The latter end of this range would typically be selected when the failure consequences are more severe, but in general, the overall geotechnical system lifetime failure probability (which includes the effects of structural and geotechnical redundancy) should be no larger than that of the supported structure. Often a foundation failure probability that is lower than that of the supported structure is selected simply because foundations are very expensive to repair, and their failure generally also leads to the additional costs associated with failure of the supported superstructure.

The level of understanding of ground conditions is reflected in this study by varying the coefficient of variation, v_ϕ, and by varying the effective distance between the pile and the sampling location, r/θ. Since the coefficient of variation, v_ϕ, and the correlation length, θ, are often unknown for a given site, a range in v_ϕ was considered, along with a worse-case value of θ, corresponding to the highest probability of failure. Three different sampling locations were investigated, with the distance between the pile location and the sampling location, r, ranging from 0 to 9 m. In general, for fixed design parameters, the probability of pile failure increases as r and v_ϕ increase. Conversely, this means that for fixed target failure probability, the geotechnical resistance factor decreases as r and v_ϕ increase.

Since measurement and model errors were not directly considered in this study, the geotechnical resistance factors recommended above should be considered to be upper bounds. The statistics of measurement errors are very difficult to determine, since the true values need to be known, and should include errors associated with transforming measurements into engineering properties (e.g., CPT observations to friction angles). Model errors involve the assessment of how accurately the "true" resistance is predicted by an equation such as eq. [6], a difficult task, since the true frictional resistance along with the true soil properties are rarely, if ever, known. As a result of the difficulties in quantifying measurement and model error, the effect of these errors on resistance factors are generally assessed through calibration with past experience and through the careful analysis of full-scale experiments. Calibration against existing Canadian codes (NBCC (National Research Council Canada 2005) and CHBDC (Canadian Standards Association 2006)) was adopted here to scale the required resistance factors so that they match what is currently used at an assumed moderate to low level of site understanding and a moderate maximum acceptable failure probability.

When confidence in the measured soil properties or in the model used is lower, the results presented here can be employed by assuming that the soil samples were taken further away from the pile centerline than they actually were (e.g., if low-quality soil samples are taken at the pile location, $r = 0$, the geotechnical resistance factor corresponding to a larger value of r, say $r = 9$ m, should be used), or by increasing the coefficient of variation, v_ϕ.

Many of the deep foundation design parameters, e.g., a, b, γ, and p, along with the mean load and friction angle, cancel out of the failure probability equation, which means that these parameters have no influence on the required geotech-

Table 6. Worst-case geotechnical resistance factors for various v_ϕ, r, and p_m, after calibration against Canadian codes (α_T/φ_{gu} = 3.30, r = 9 m, $p_m = 10^{-4}$).

		Geotechnical resistance factor			
r (m)	v_ϕ	$p_m = 10^{-2}$	$p_m = 10^{-3}$	$p_m = 10^{-4}$	$p_m = 10^{-5}$
0.0	0.1	0.74	0.67	0.62	0.57
0.0	0.2	0.74	0.66	0.61	0.57
0.0	0.3	0.73	0.65	0.61	0.57
0.0	0.344	0.72	0.64	0.60	0.56
4.5	0.1	0.73	0.66	0.60	0.56
4.5	0.2	0.71	0.63	0.57	0.52
4.5	0.3	0.65	0.55	0.49	0.44
4.5	0.344	0.62	0.52	0.44	0.39
9.0	0.1	0.73	0.66	0.60	0.55
9.0	0.2	0.70	0.61	0.55	0.50
9.0	0.3	0.63	0.52	0.46	0.40
9.0	0.344	0.57	0.47	0.40	0.35

nical resistance factor, according to this theory. Since the geotechnical resistance factor is primarily aimed at accounting for geotechnical variability (i.e., coefficient of variation, v_ϕ), it makes sense that it should be largely independent of fixed and known design parameters.

However, this does not mean that errors made in estimating any or all of these design parameters (a, b, γ, p, and the means of the loads and resistances) will have no affect on the true probability of pile failure. These parameters disappear from the proposed failure probability calculation only because the assumed true resistance given by eq. [6] makes use of the very same parameters used in the design calculations of eq. [9]. In other words, if the design parameters are in error, the true resistance will be correspondingly in error and the failure probability ends up comparing two random quantities (load versus resistance), both of which are aimed at correspondingly wrong quantities.

A solution to the fundamental issue of how to handle errors in design parameters is to calibrate the resistance factors against values that are currently acceptable and based on years of experience with real “data” (fraction of designed piles that fail in practice), which of course makes sense in any case, as mentioned above. What this means is that the resistance factors proposed in this paper should really be used in a comparative sense, i.e., as a way to determine how existing resistance factors should be changed with respect to changes in site variability, understanding, and maximum tolerable failure probability (dependent on failure consequences). For example, suppose that the current code specified geotechnical resistance factor is 0.4, and that this is based on a medium to low level of site understanding (equivalent to, say, r = 9 m), is aimed at a lifetime failure probability of 10^{-4}, and employs the same load factors as used herein. The corresponding geotechnical resistance factor proposed in row five of Table 5 is quite a bit higher, at φ_{gu} = 0.65. This discrepancy would almost certainly be due to omitting measurement and model errors, but may also be suggesting that the current resistance factor is overly conservative. Both possibilities would need further investigation. A first pass calibration would involve scaling all of the φ_{gu} values in the first six rows of Table 5 by 0.40/0.65 so that the calibrated resistance factors would now range from 0.52 (row one) to 0.35 (row six) in Table 5. A second pass calibration would involve explicitly adding an error term to the true resistance, with variance selected so as to achieve the current geotechnical resistance factor of 0.4.

The relative agreement between the theoretically derived geotechnical resistance factors proposed in this study and those used in the 1992 CFEM (Canadian Geotechnical Society 1992), and to a lesser extent with the lower end of AS 5100.3 (Standards Australia 2004), as shown in Table 5, is, however, encouraging. The current study provides a rigorous mathematical basis for the determination of geotechnical resistance factors in pile design in soils under effective stress (frictional) conditions, and the theory provides a framework to extend code provisions beyond only calibration with experiment and the past.

Acknowledgements

The authors would like to thank the Natural Sciences and Engineering Research Council of Canada (NSERC), under Discovery Grant OPG0105445, for their essential support of this research. The authors would also like to thank Farzaneh Naghibi for her help with the tables and with reviewing the work.

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List of symbols

a earth pressure coefficient
b pile interface friction angle coefficient
D depth of soil sample
E expectation operator
d_1 first derivative of $\ln\hat{X}$ and $\ln\overline{X}$
d_2 second derivative of $\ln\hat{X}$ and $\ln\overline{X}$
d_3 third derivative of $\ln\hat{X}$ and $\ln\overline{X}$
F total true (random) load
F_{D} true (random) dead load
$\hat{F}_{\mathrm{D}}$ characteristic dead load ($\hat{F}_{\mathrm{D}} = k_{\mathrm{D}}\mu_{\mathrm{D}}$)
$\hat{F}_i$ ith characteristic load effect
F_{L} true (random) live load
$\hat{F}_{\mathrm{L}}$ characteristic live load ($\hat{F}_{\mathrm{L}} = k_{\mathrm{L}}\mu_{\mathrm{L}}$)
f_Y probability density function of Y
f_ϕ probability density function of friction angle
G_ϕ standard normal random field of friction angle
H designed pile length
I_i importance factor corresponding to ith characteristic load effect
m number of soil observations
n number of elements in pile
n_{f} number of times Monte Carlo simulation steps result in pile failure
n_{sim} number of times Monte Carlo simulation steps are repeated
P probability operator
p pile perimeter length
p_{f} probability of failure
p_{m} maximum acceptable probability of failure
q factored design load $= \alpha_{\mathrm{L}}\hat{F}_{\mathrm{L}} + \alpha_{\mathrm{D}}\hat{F}_{\mathrm{D}}$
$R_{\mathrm{D/L}}$ ratio of dead to live load means
$\hat{R}_{\mathrm{D/L}}$ characteristic dead to live load ratio
R_{u} true ultimate resistance (random)
$\hat{R}_{\mathrm{u}}$ ultimate characteristic resistance (based on characteristic soil properties)
r distance between soil sample and pile centerline
s scale factor governing friction angle variability
t distance between two points
v_ϕ coefficient of variation of friction angle
$\hat{X}$ defined by eq. [29*a*]: $\hat{X} = (1 - \sin\hat{\phi})\tan(b\hat{\phi})$
$\overline{X}$ defined by eq. [29*b*]: $\overline{X} = (1 - \sin\overline{\phi})\tan(b\overline{\phi})$
Y defined by eq. [29*d*]: $Y = F\hat{X}/\overline{X}$
y a realization of Y
Z vertical coordinate axis
z depth from ground surface
Δz vertical dimension of soil samples
z_1 depth of point 1
z_2 depth of point 2
z_i depth of point i
z_i^{o} depth of the center of the ith soil sample
α_{D} dead load factor
α_i load factor corresponding to the ith load effect
α_{L} live load factor
α_{T} total load factor
γ effective unit weight of soil
γ_{HD} average correlation coefficient between the friction angle samples over length D and the friction angle along the pile of length H
$\gamma(H)$ variance function giving variance reduction due to averaging over pile length H
$\delta(z)$ average interface friction angle between the soil and the pile perimeter at depth z
θ correlation length of the random friction angle field, G_ϕ
μ_{D} mean dead load
μ_{F} mean total load on pile
μ_{L} mean live load
$\mu_{\ln F}$ mean total log load on pile
$\mu_{\ln\hat{X}}$ mean of $\ln\hat{X}$
$\mu_{\ln\overline{X}}$ mean of $\ln\overline{X}$
$\mu_{\ln Y}$ mean of $\ln Y$
μ_ϕ mean friction angle
$\mu_{\hat{\phi}}$ mean of the characteristic friction angle (based on an arithmetic average of friction angle observations)
$\mu_{\overline{\phi}}$ mean of the equivalent friction angle (based on an arithmetic average of friction angle over pile length H)
ρ correlation coefficient
σ_ϕ friction angle standard deviation
σ_{D} dead load standard deviation
σ_{F} total load standard deviation
σ_{L} live load standard deviation
$\sigma_{\ln F}$ standard deviation of total log load
$\sigma_{\ln Y}$ standard deviation of $\ln Y$
$\sigma_{\ln\hat{X}}$ standard deviation of $\ln\hat{X}$
$\sigma_{\ln\overline{X}}$ standard deviation of $\ln\overline{X}$
$\sigma_{\hat{p}_{\mathrm{f}}}$ standard deviation of failure probability estimate
σ_ϕ friction angle standard deviation
$\sigma_{\hat{\phi}}$ standard deviation of $\hat{\phi}$
$\sigma_{\overline{\phi}}$ standard deviation of $\overline{\phi}$
$\sigma'_{\mathrm{o}}(z)$ effective vertical stress at depth z
$\tau(z)$ average ultimate shear stress acting on the perimeter of the pile at depth z
Φ standard normal cumulative distribution function
ϕ friction angle
$\hat{\phi}$ characteristic value of friction angle
$\overline{\phi}$ equivalent friction angle of the soil

φ_{gu} ultimate geotechnical resistance factor
$\hat{\phi}_i$ arithmetic average of observed friction angles (i = 1, 2, …, m)
$\overline{\phi}_i$ local average of $\phi(z)$ over the ith element (i = 1, …, n)
ϕ_{max} maximum friction angles in radians
ϕ_{min} minimum friction angles in radians
$\phi(z)$ average effective angle of internal friction of the soil around the pile perimeter at depth z

Appendix A

The first, second, and third derivatives of $\ln\overline{X}$ and $\ln\hat{X}$ are given by d_1, d_2, and d_3, respectively, as follows. The derivates are evaluated at the mean. See Naghibi (2010) for more details.

$$[\text{A1}a] \qquad d_1 = \left.\frac{\partial \ln\hat{X}}{\partial\hat{\phi}}\right|_{\mu_\phi} = \left.\frac{\partial \ln\overline{X}}{\partial\overline{\phi}}\right|_{\mu_\phi} = \frac{\cos(\mu_\phi)}{[\sin(\mu_\phi)-1]} + \frac{2b}{\sin(2b\mu_\phi)}$$

$$[\text{A1}b] \qquad d_2 = \left.\frac{\partial^2 \ln\hat{X}}{\partial\hat{\phi}^2}\right|_{\mu_\phi} = \left.\frac{\partial^2 \ln\overline{X}}{\partial\overline{\phi}^2}\right|_{\mu_\phi} = \frac{1}{[\sin(\mu_\phi)-1]} - \frac{4b^2\cos(2b\mu_\phi)}{\sin^2(2b\mu_\phi)}$$

$$[\text{A1}c] \qquad d_3 = \left.\frac{\partial^3 \ln\hat{X}}{\partial\hat{\phi}^3}\right|_{\mu_\phi} = \left.\frac{\partial^3 \ln\overline{X}}{\partial\overline{\phi}^3}\right|_{\mu_\phi} = \frac{-\cos(\mu_\phi)}{[1-\sin(\mu_\phi)]^2} + \frac{8b^3}{\sin(2b\mu_\phi)} + \frac{2\cos^2(2b\mu_\phi)}{\sin^3(2b\mu_\phi)p}\pi$$

Reference

Naghibi, M. 2010. Geotechnical resistance factors for ultimate limit state design of deep foundations. Ph.D. thesis, Dalhousie University, Engineering Mathematics, Halifax, N.S.

List of symbols

b pile interface friction angle coefficient
d_1 first derivative of $\ln\hat{X}$ and $\ln\overline{X}$
d_2 second derivative of $\ln\hat{X}$ and $\ln\overline{X}$
d_3 third derivative of $\ln\hat{X}$ and $\ln\overline{X}$
p pile perimeter length
$\hat{X}$ defined by eq. [29a]: $\hat{X} = (1-\sin\hat{\phi})\tan(b\hat{\phi})$
$\overline{X}$ defined by eq. [29b]: $\overline{X} = (1-\sin\overline{\phi})\tan(b\overline{\phi})$
μ_ϕ mean friction angle
$\hat{\phi}$ characteristic value of friction angle
$\overline{\phi}$ equivalent friction angle of the soil