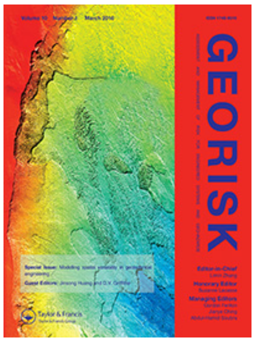

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Influence of soil shear strength spatial variability on the compressive strength of a block

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ABSTRACT
This paper studies the probability of failure of a soil block consisting of random soil using the random finite element method (RFEM). The shear strength parameters (c', $\tan\phi'$) are treated as random variables characterized by a mean, a coefficient of variation and a spatial correlation length θ. Both normal and log-normal input distributions are considered. A "worst-case" spatial correlation length, which leads to a maximum probability of failure, is clearly demonstrated. This "worst-case" spatial correlation length, which is also observed in other geotechnical applications, has implications for design and can be used in the absence of more detailed information about the soil spatial variability to target a conservative design.

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KEYWORDS
Random finite element method; spatial variability; worst-case spatial correlation length; block compressive strength; probabilistic analysis

Notation

B	block height
c'	soil cohesion
E	Young modulus
FS	factor of safety
p_f	probability of failure
q_{det}	deterministic compressive strength based on the mean of input variables
q_{Median}	deterministic compressive strength based on the median of input variables
q_u	compressive strength
V	coefficient of variation
θ	spatial correlation length
Θ	dimensionless spatial correlation length
μ	mean
υ	Poisson's ratio
ρ	cross-correlation between random variables
ρ_τ	point-wise correlation in random field
σ	standard deviation
τ	vector between two points in a random field
φ'	friction angle of soil

Introduction

The paper uses the random finite element method (RFEM; e.g. Griffiths and Fenton 1993) to investigate the compressive strength statistics of a block of soil with random cohesion and friction. There are several tools for probabilistic analysis in geotechnical engineering, but few of them properly take account of soil spatial variability. The spatial variability of soil properties has shown to affect the behaviour of geotechnical problems such as bearing capacity, seepage, settlement, shear strength and slope stability analysis (e.g. Griffiths and Fenton 2004; Soubra, Massih, and Kalfa 2008; Griffiths, Huang, and Fenton 2009; Cho and Park 2010; Huang, Griffiths, and Fenton 2010; Ching and Phoon 2013; Ching, Phoon, and Kao 2014). Other studies have claimed that proper spatial correlation modelling is unnecessary because the highest probability of failure corresponds to that with an infinite spatial correlation length, that is, that obtained using a single random variable analysis in which each simulation is uniform (e.g. Cho 2007; Javankhoshdel and Bathurst 2014). The block compression problem was chosen in this paper as a simple problem that demonstrates phenomena, such as the "worst-case" spatial correlation length, that can be extrapolated to other geotechnical application. The paper focuses on the influence of (1) the spatial correlation length of shear strength parameters and (2) the input probability density functions, that is, normal and log-normal.

RFEM model

As shown in Figure 1, a plane strain square block of soil with the boundary conditions as indicated was subjected to fixed incremental vertical displacement on the top surface. The mesh consisted of square eight-node elements.

The RFEM implementation used in this study combines elastic–plastic finite element analysis (e.g. Smith and Griffiths 2004) with random fields generated using the local average subdivision (LAS) method (Fenton and Vanmarcke 1990). This method is described in detail

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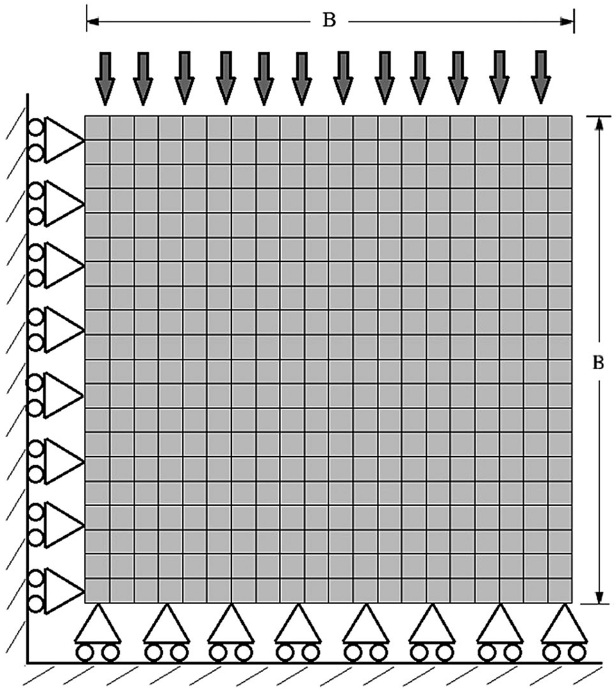

Figure 1. Loading and boundary conditions for the block compression problem.

in Fenton and Griffiths (2008). A benefit of the RFEM is that full account is taken of local averaging and variance reduction over each element. The RFEM is used in conjunction with Monte-Carlo simulations in which the analysis is repeated until the probabilities relating to output quantities of interest become statistically reproducible. For each Monte-Carlo simulation, the top nodes of the block were incrementally displaced vertically, until the sum of the nodal reactions divided by block width B levelled out to within a given tolerance. At this point, the compressive strength q_u was recorded before moving on to the next simulation. Although the soil properties generated for each simulation of the Monte-Carlo process involve the same mean, standard deviation and spatial correlation length, the spatial distribution of properties varies from one realization to the next, hence each simulation gives a different value of q_u. Following a sufficient number of realizations, the mean and standard deviation of the compressive strength q_u can be computed.

The spatial correlation length has units of length and represents the distance over which the soil property in question is reasonably well correlated to its neighbours. In this paper, a "Markovian" correlation function is used where the spatial correlation is assumed to decay exponentially with distance (Vanmarcke 1984)

$$\rho_\tau = e^{-2\tau/\theta} \tag{1}$$

where θ is the isotropic spatial correlation length and ρ_τ is the correlation coefficient between any two points separated by τ. In this study, the spatial correlation length is non-dimensionalized in the form $\Theta = \theta/B$.

The soil cohesion c' and tangent of internal friction angle $\tan \varphi'$ were modelled as random variables, while Young's modulus E and Poisson's ratio υ were deterministic and held constant at 10^5 kN/m^2 and 0.3, respectively. Both normal and log-normal distributions of random variables c' and $\tan \varphi'$ were considered separately in this study. Both distributions were defined by a mean μ and a standard deviation σ. In the normal case, the probability density function of (shown here for c') was given by

$$f(c') = \frac{1}{\sigma_{c'}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{c' - \mu_{c'}}{\sigma_{c'}}\right)^2\right\} \tag{2}$$

and in log-normal case by

$$f(c') = \frac{1}{c' \ \sigma_{\ln c'}\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\ln c' - \mu_{\ln c'}}{\sigma_{\ln c'}}\right)^2\right\} \tag{3}$$

The coefficient of variation V and spatial correlation length Θ were assumed to be the same for both c' and $\tan \phi'$, thus

$$V = \frac{\sigma_{c'}}{\mu_{c'}} = \frac{\sigma_{\tan \phi'}}{\mu_{\tan \phi'}} \tag{4}$$

and

$$\Theta = \Theta_{c'} = \Theta_{\tan \phi'} \tag{5}$$

In order to maintain reasonable accuracy and run-time efficiency, the influence of mesh refinement was examined for two different mesh densities as shown in Figure 2, to decide on the optimum meshing for the model. The mesh density has been shown to have a significant influence in finite element analysis (e.g. Huang and Griffiths 2015). The "coarse" mesh shown in Figure 2(a) had 400 elements while the "fine" mesh shown in Figure 2(b) had 1600 elements. A series of analyses was then performed in which the mean of c' and $\tan \varphi'$ were kept constant and equal to $\mu_{c'} = 100$ kPa and $\mu_{\tan \varphi'} = \tan 30° = 0.577$. The coefficient of variation was fixed at $V = 0.3$ for both random variables, while the (isotropic) spatial correlation length Θ was varied. The variables c' and $\tan \phi'$ were initially assumed to be uncorrelated.

The mean compressive strength μ_{qu} was calculated by averaging the compressive strength over 2000 Monte-Carlo simulations and normalizing it with respect to the deterministic value $q_{\mathrm{det}} = 346.4$ kPa, which was based on the mean values of input parameters $\mu_{c'}$ and

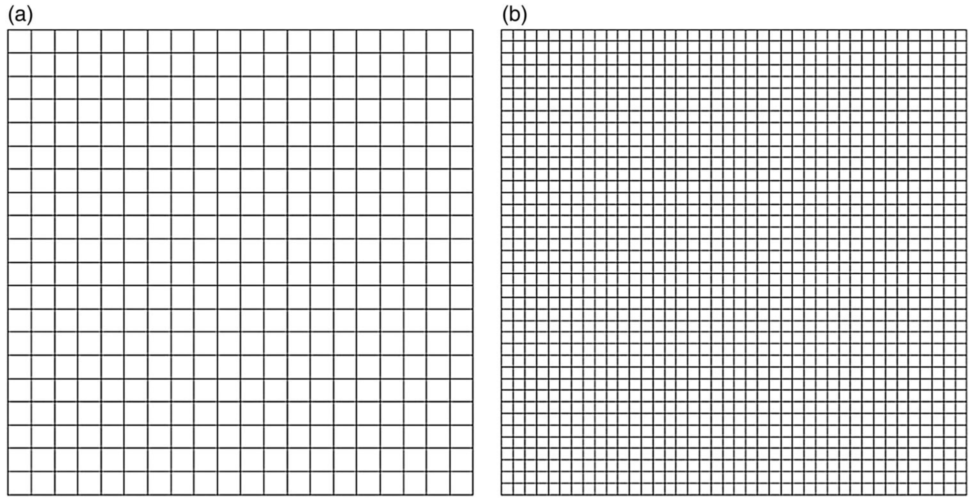

Figure 2. Mesh used for the finite element block compression analysis.

$\mu_{\tan\varphi'}$ using the equation (Griffiths, Fenton, and Tveten 2002):

$$q_{\det} = 2\mu_{c'}\left[\mu_{\tan\phi'} + (1+\mu^2_{\tan\phi'})^{(1/2)}\right] \quad (6)$$

Figure 3 shows the effect of mesh density on the normalized compressive strength ($\mu_{qu}/q_{\det}$) versus spatial correlation length (Θ) for the block with $V = 0.3$. A log-normal distribution was assumed for the input random variables. The finer mesh indicated a slightly lower mean strength since it gives more failure path options through the block. The difference is quite small however, so the coarser mesh is deemed to give reasonable precision for the current paper.

Figure 4 shows two typical failure mechanisms with the soil cohesion distribution in the form of a grey scale in which weaker regions are lighter and stronger regions are darker. Figure 4(a) has a relatively low spatial correlation length while Figure 4(b) presents a model with higher spatial correlation length. It should be emphasized that the mean and standard deviation of the input random variables are the same in both cases.

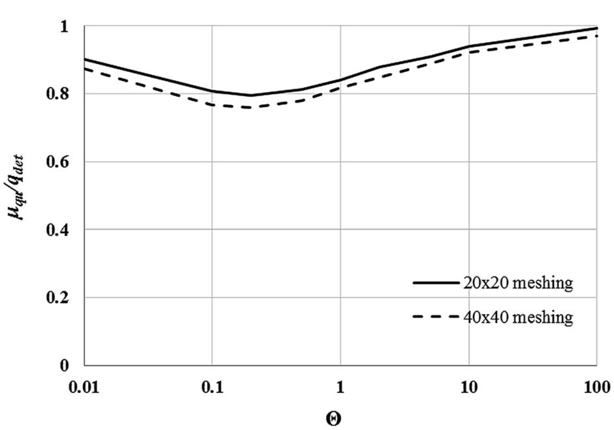

Figure 3. Variation of μ_{qu} with Θ with different mesh densities, $V = 0.3$.

Parametric study

The same block with $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$ was then analysed with different coefficients of variation and spatial correlation lengths. Figure 5 illustrates the results of these analyses with log-normal input random variables.

An observation from Figure 5 is that by increasing Θ, the mean compressive strength of the block converges to the deterministic value based on the mean input variables for all cases. There are considerable differences at lower values of the spatial correlation length however.

When Θ approaches 0, the average of the random field tends to the median (e.g. Griffiths and Fenton 2004), hence the compressive strength can be estimated analytically from Equation (6) using the median values of the input random variables. As shown below, when $V = 0.3$ the median of the shear strength parameters leads to a normalized compressive strength of 0.94, which is in close agreement with Figure 5. Similar checks could be made for other values of V

$$\text{Median}_{c'} = \frac{\mu_{c'}}{\sqrt{1+V^2}} = \frac{100}{\sqrt{1+0.3^2}} = 95.78\,\text{kPa}$$

$$\text{Median}_{\tan\phi'} = \frac{\mu_{\tan\phi'}}{\sqrt{1+V^2}} = \frac{0.577}{\sqrt{1+0.3^2}} = 0.553$$

$$q_{\text{Median}} = 2c'\left[\tan\phi' + (1+\tan^2\phi')^{0.5}\right] = 2\times 95.78\,[0.553 + (1+0.553^2)^{0.5} = 324.8\,\text{kPa}$$

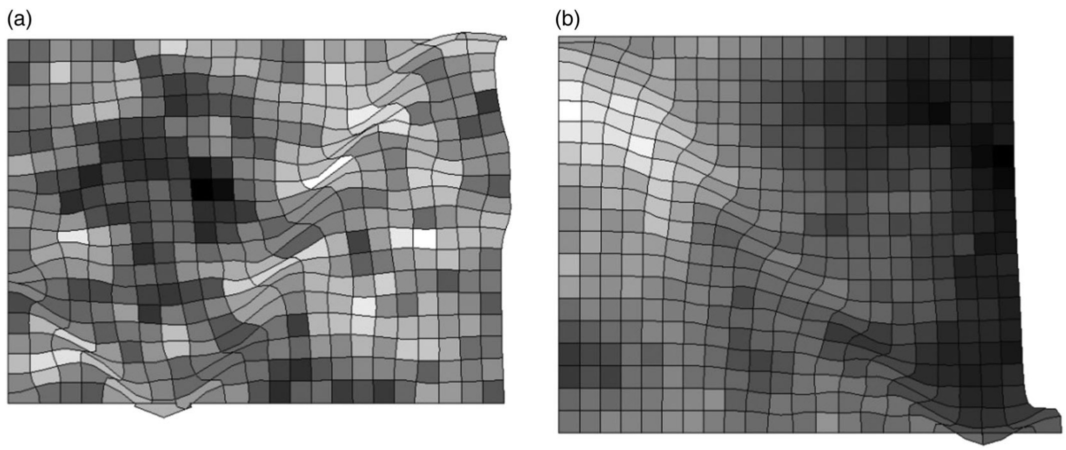

Figure 4. Typical random field realization and failure mechanisms for the block compression problem, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$: (a) $\Theta = 0.1$ and (b) $\Theta = 5$. (The deformed meshes at the bottom of the models are because of graphical scaling.)

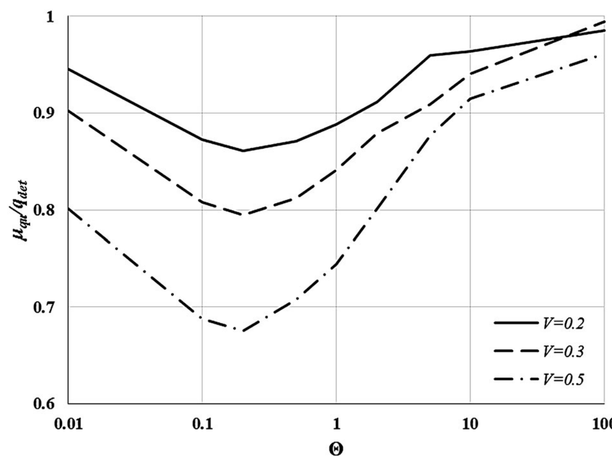

Figure 5. Variation of normalized μ_{qu} with Θ and V with a log-normal distribution for the input random variables, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$.

$$\frac{q_{\text{Median}}}{q_{\text{det}}} = \frac{324.8}{346.5} = 0.937$$

The normalized compressive strength displays a minimum value for all cases when Θ between 0.1 and 1.

Figure 6 shows results from a similar parametric study using normal distributions for c' and $\tan\phi'$. The trends are similar to that obtained with the log-normal distribution, however, the "worst-case" phenomenon is more pronounced. For example, with $V = 0.5$ the minimum value of the normalized compressive strength is about 0.4 compared with 0.68 in the log-normal case. For a better comparison, the normalized compressive strength versus Θ curves for both normal and log-normal distributions have been plotted on the same graph for $V = 0.2$, 0.3 and 0.5 in Figure 7.

It was speculated that the reason for the difference between normal and log-normal results might lie in the occasional generation of negative property values when using the normal distribution. For example, for $V = 0.3$, 0.04% of each input values could be negative, and when $V = 0.5$, this number increases to 2.27%. These percentages remain quite small, however, and cannot explain the differences shown in Figure 7. A more likely explanation may be related to the overall difference between normal and log-normal distributions as V is increased as shown in Figure 8. For $V = 0.2$, the distributions are quite similar but are starting to differ significantly for $V = 0.5$.

Probability of failure (p_f)

A probability of failure p_f in the block problem can be defined as the probability that the true compressive

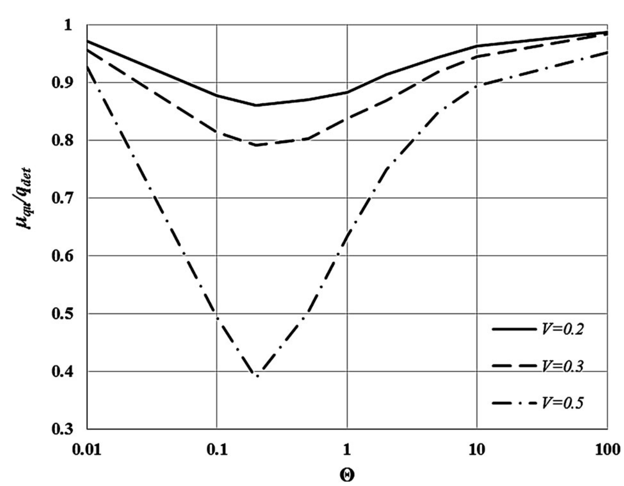

Figure 6. Variation of normalized μ_{qu} with Θ and V with normal distribution for the input random variables, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$.

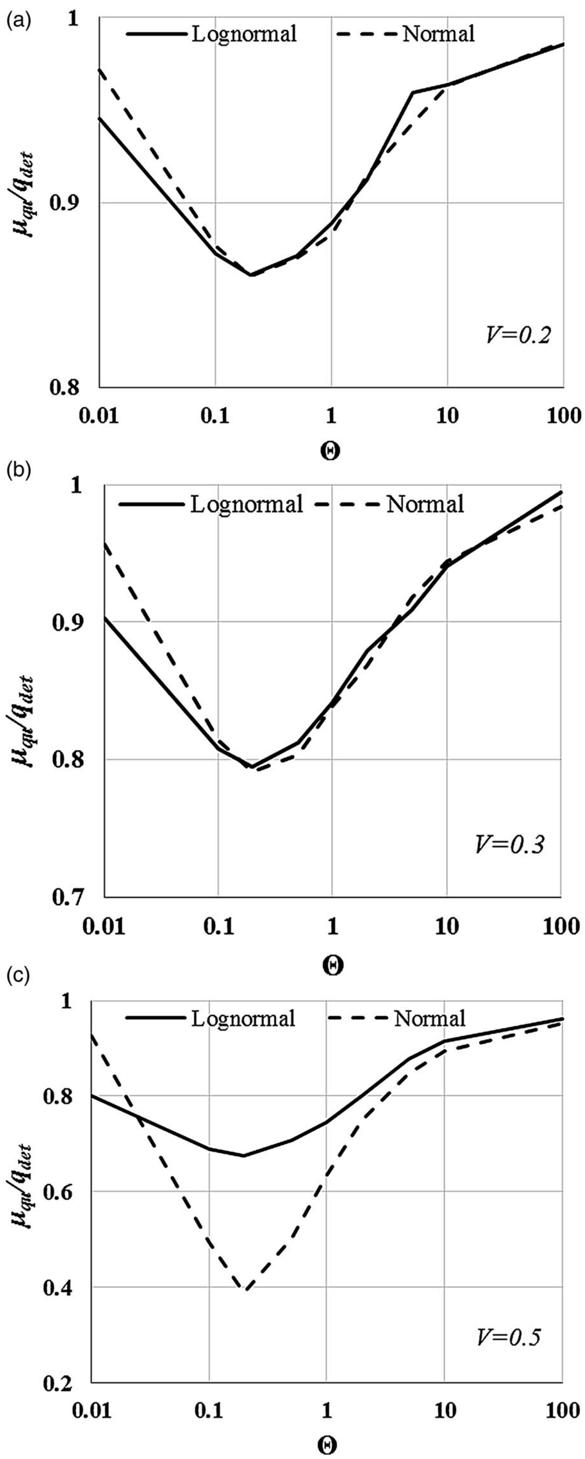

Figure 7. Variation of normalized μ_{qu} with Θ with normal and log-normal distributions for the input random variables, $\mu_{c'} =$ 100 kPa and $\mu_{\tan\varphi'} = 0.577$, (a) $V = 0.2$, (b) $V = 0.3$ and (c) $V = 0.5$.

strength (calculated with RFEM) is less than factored strength calculated with mean values of c' and tan φ', that is,

$$p_f = P\left[q_u < \frac{q_{\text{det}}}{\text{FS}}\right] \quad (7)$$

The influence of the cross-correlation ρ between random variable c' and tan φ' was studied by considering values in the range $0 \le \rho \le 1$, where $\rho = 1$ gives perfect positive correlation, $\rho = 0$ gives no correlation, and $\rho = -1$ gives perfect negative correlation. The p_f variation with the normalized spatial correlation length Θ for these three analyses have been plotted in Figure 9. The results were obtained with log-normal distribution functions for both variables and a coefficient of variation equal to 0.2. A factor of safety of 1.3 was applied for calculating the probability of failure.

Figure 9 demonstrates very significant differences in p_f depending on the value of the cross-correlation coefficient between c' and tan φ'. Quite high p_f values, up to about 0.25, are observed when $\rho = 1$, while the probability of failure when $\rho = -1$ is essentially zero for all values of spatial correlation. Zero correlation with $\rho = 0$ leads to intermediate (although still quite high) values of p_f. Similar trends have been reported for other geotechnical engineering problems such as slope stability analysis (e.g. Allahverdizadeh, Griffiths, and Fenton 2015). Although the assumption of positive cross-correlation is the most conservative, it has been suggested by some investigators (e.g. Cherubini 2000) that c' and tan φ' may actually be negatively cross correlated. Thus, modelling the block with no correlation ($\rho = 0$) between c' and tan φ' might still be on the conservative side. Cross-correlation between c' and tan ϕ' is a topic of need of further investigation (e.g. Griffiths, Huang, and Fenton 2009), but for the remainder of this study, the shear strength parameters will be assumed to be uncorrelated.

Figure 10 shows the p_f versus Θ for the block when $V = 0.2$ for different factors of safety when the input random variables have a log-normal distribution. By increasing FS, the p_f decreases, but there is still 5% probability of failure at high spatial correlation lengths with FS = 1.5. The "worst-case" observation made earlier in this paper now manifests itself as a maximum probability of failure. The "worst-case" spatial correlation length Θ_w is about 0.5 when FS is equal to 1.2 and increases as FS increases. When FS = 1.5, Θ_w is essentially tending to infinity. This "worst-case" phenomenon is an important observation that suggests that single random variable approaches that ignore spatial variability (by essentially assuming $\Theta \to \infty$) can be unconservative. The "worst-case" spatial correlation length was observed previously in bearing capacity and settlement analysis of strip footings (e.g. Ahmed and Soubra 2012; Al-Bittar and Soubra 2013) and slope stability analysis (e.g. Allahverdizadeh, Griffiths, and Fenton 2015).

When Θ is small, c' and tan φ' are highly spatially variable, hence each Monte-Carlo simulation gives

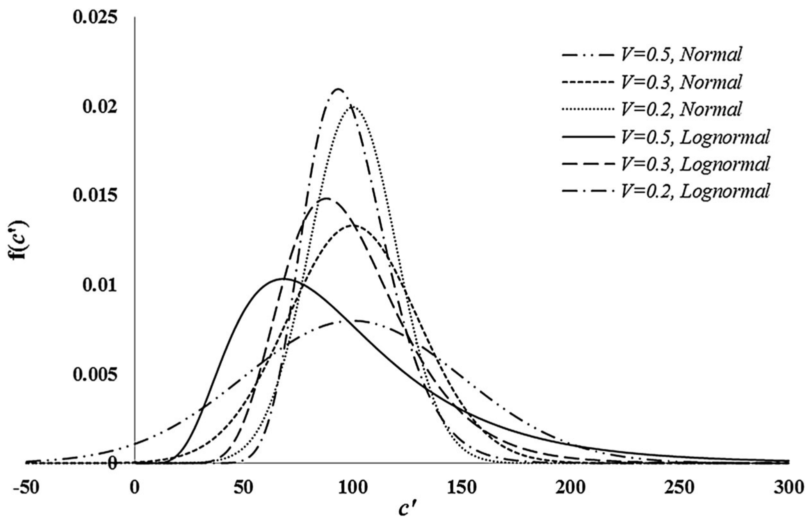

Figure 8. Normal and log-normal distributions of c' with different coefficients of variation, $\mu_{c'} = 100$ kPa.

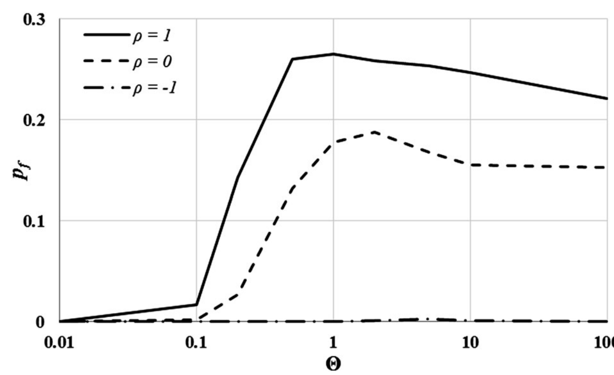

Figure 9. Variation of the probability of failure with Θ for different cross-correlations, $V = 0.2$, FS = 1.3, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$.

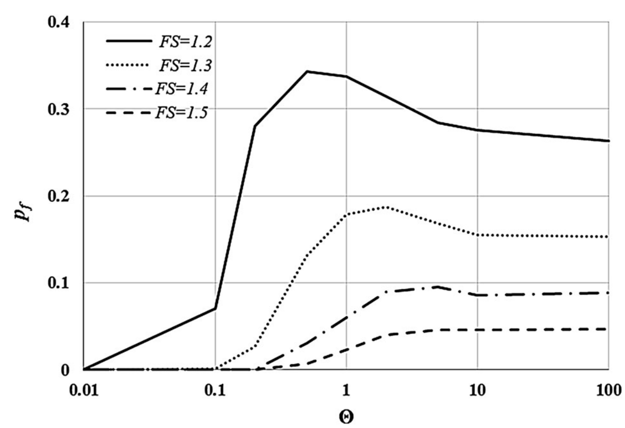

Figure 10. Variation of the p_f with FS and Θ, $V = 0.2$ with log-normal distribution, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$.

essentially the same result. As mentioned previously, when using a log-normal distribution the average of the soil shear strength parameters tends to be the median. Considering the case of $V = 0.2$, the median of c' and $\tan\varphi'$ is 98.06 and 0.566, respectively; thus, the compressive strength of the block based on the median of input random variables from Equation (6) is $q_{\text{Median},\ V=0.2} = 336.35$ kPa.

This value is less than the $q_{\text{det}} = 346.5$ kPa; however, by applying a factor of safety of 1.2 (say) the factored q_{det} becomes 288.75 kPa implying a probability of failure of 0. If the factored value of q_{det} was still higher than the q_{Median}, a probability of failure of 1 would appear for that case. This case is more likely to happen when the coefficient of variation of random variables increases which results in lower median values for random variables.

For large spatial correlation lengths (e.g. $\Theta \approx 100$), each simulation is essentially uniform but different from one simulation to the next. Thus, the value of the p_f in this case agrees with that obtained using Monte-Carlo with one random variable approach as shown in Table 1. These predicted values are very close to the ones observed in Figure 10 when $\Theta = 100$. The only case that the p_f at $\Theta = 100$ is slightly higher than its value in Table 1 is when FS = 1.2. The reason is that for this case p_f is still decreasing by increasing Θ from 10 to 100. By approaching Θ to infinity, this value is expected to converge to the values presented in Table 1.

Table 1. p_f values for the block compression problem using a single random variable approach $\Theta \to \infty$ with $V = 0.2$, $\mu_{c'} = 100$ kPa and $\mu_{\tan\varphi'} = 0.577$.

Factor of safety	Probability of failure
1.2	0.236
1.3	0.137
1.4	0.079
1.5	0.039

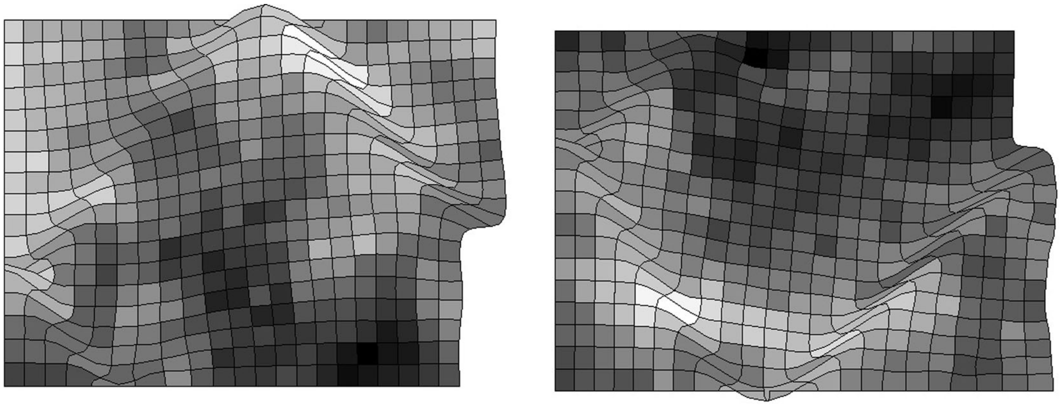

Figure 11. Two typical failure mechanism for the block compression problem when $V = 0.2$ and $\Theta = \Theta_w = 1$. (The deformed meshes at the top and bottom of the models are because of graphical scaling.)

In finite element analyses, the failure mechanism is free to seek out the weakest path through the soil. Figure 11 shows two typical failure mechanisms for the block compression problem when $V = 0.2$ and the spatial correlation length of the random variable is set $\Theta = 1$, which is close to the "worst-case" value. The reason for the "worst-case" phenomenon is thought to be that there is some intermediate spatial correlation lengths that facilitate more failure options over a set of Monte-Carlo simulations. The two examples in Figure 11 show the failure mechanism going through weak regions, (represented by lighter shades), and developing failure modes that could not happen in a more uniform block.

The p_f versus Θ for different values of V has been plotted in Figure 12 when FS = 1.3. This figure shows that the probability of failure also increases by increasing the coefficient of variation. This graph has two branches: one when $V \leq 0.4$ where p_f starts at 0 and subsequently increases with Θ and another when $V \geq 0.5$ where p_f starts at 1 and subsequently decreases. The reason is that the coefficient of variation of $V = 0.5$ results in lower values for q_{median} than the factored q_{det} as it was explained earlier.

As shown in Figure 12, the coefficient of variation also seems to have an influence on the worst-case spatial correlation length, with Θ_w increasing as V is decreased. For this block compression problem when FS = 1.3, $V = 0.5$ gives $\Theta_w \approx 0.01$ and when $V = 0.2$, $\Theta_w \approx 2$.

The probability of failure has also been calculated for the block with normally distributed input random variables as shown in Figure 13 when FS = 1.3. Log-normal and normal are compared directly for $V = 0.2$ in Figure 14 and show similar results.

Comparison of Figures 12 and 13 shows that log-normal and normal distributions for the input random variables lead to similar p_f for the block compression problem when the coefficient of variation is less than 0.4. For the case with $V = 0.5$, however, there is a very significant difference between the normal and log-normal distributions.

When Θ is small, the block becomes increasingly uniform with essentially constant strength at each

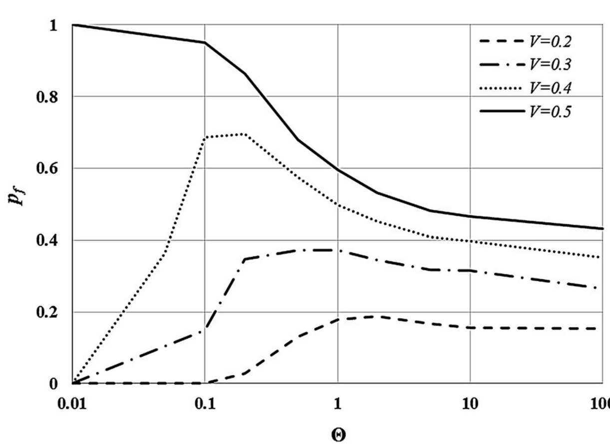

Figure 12. Variation of p_f with Θ and V, FS = 1.3 with log-normal distribution, $\mu_{c'} = 100$ kPa and $\mu_{\tan\,\varphi'} = 0.577$.

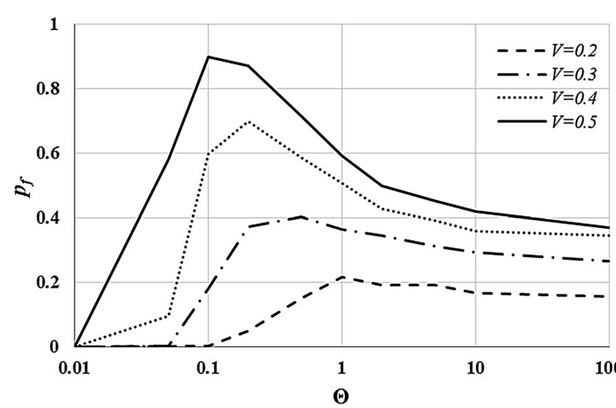

Figure 13. Variation of p_f with Θ and V, FS = 1.3 with normal distribution, $\mu_{c'} = 100$ kPa and $\mu_{\tan\,\varphi'} = 0.577$.

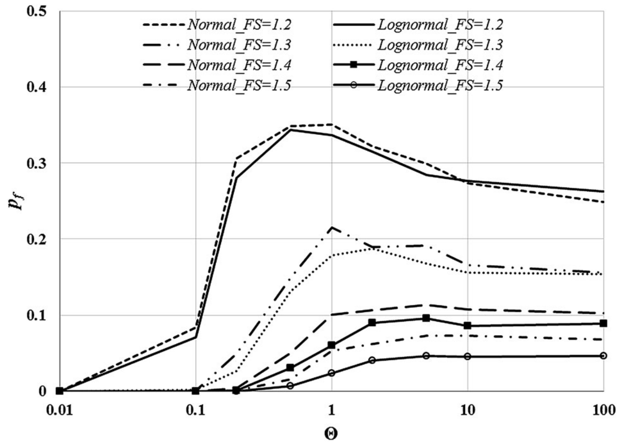

Figure 14. Variation of p_f with FS and Θ, $V = 0.2$ with normal and log-normal distributions.

simulation. For the block model with the normal distribution, c' and tan φ' tend to their mean values; however, in the log-normal case, they tend to their median values which are lower (weaker). The mean values of c' and tan φ' in the normal case give the same values for the compressive strength as q_{det} and consequently $p_f = 0$ after application of any factor of safety >1. In the log-normal case using median values, the factored compressive strength was calculated to be less than the q_{det} when $V = 0.5$, hence $p_f = 1$.

Conclusions

This paper has studied the probability of failure for a block compression problem using the RFEM. The influence of the coefficient of variation V, spatial correlation length Θ, cross-correlation between strength parameters and the choice of input random variable distribution functions (normal or log-normal) on the probability of failure p_f were investigated. It was shown clearly that a worst-case spatial correlation length exists for the block compression problem studied in this paper. The worst-case spatial correlation length, leading to a maximum probability of failure, was shown to be of the order of $0.1B$ to $2B$, where B is the square block dimension. The implication of this result for the design is that in the absence of good quality data on the spatial correlation length, it should be fixed to its worst-case value Θ_w in the interests of conservatism. This result is a practical and important finding, as the soil spatial variability is generally difficult and expensive to estimate accurately.

A brief investigation of cross-correlation between c' and tan ϕ' indicated that the level of correlation could make a significant difference to the probability of failure for a given factor of safety. The results show that a negative cross-correlation between the two random variables considered in this paper, can result in a smaller probability of failure than the case with no cross-correlation, while considering a positive cross-correlation would increase the probability of failure.

For the block problem, results from normal and log-normal input distributions were similar when the coefficient of variation was small and increasingly differed as the coefficient of variation increases. The biggest differences between the results of normal and log-normal distributions occurred at low spatial correlation lengths, when local averaging can result in properties being "safe" with the normal distribution and "unsafe" with the log-normal. It should be remembered, however, that the results of the RFEM start to display discretization errors and statistical distortion due to local averaging when the elements are too big to properly model the spatial variability. It can still be stated, however, that careful consideration should always be given to the choice of input probability density functions for geotechnical analysis. Distributions should be based on high-quality field data if available, but if that is not available, a conservative approach should always be followed.

Disclosure statement

No potential conflict of interest was reported by the authors.

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