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Research Paper

On a unified theory for reliability-based geotechnical design

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ABSTRACT

Some theoretical models have been developed over the years by the authors dealing with reliability-based design of geotechnical systems. A comparison reveals that these models follow the same form which can be used to develop a unified reliability-based model, which includes the effects of spatial variability, site understanding, and failure consequence severity. This paper describes the unified model and applies it to four geotechnical problems to determine resistance and consequence factors to be used in design. The problems considered are the ultimate and serviceability limit state design of shallow foundations and ultimate and serviceability limit state design of deep foundations.

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1. Introduction

By and large, the ground is one of the most highly variable, hence uncertain, engineering materials. Unlike quality controlled materials such as wood, concrete, or steel, whose probability distributions are well known and relatively constant world-wide, geotechnical designers face large resistance uncertainties from site to site, and even within a site. Because of this site specific uncertainty, there is a real desire in the geotechnical community to account for site understanding in order to achieve economical, yet safe designs. This can be achieved by adjusting the resistance factor (or factored resistance) as a function of site and model understanding, allowing overall safety to be maintained at a common target maximum failure probability as well as demonstrating the direct economic advantage of increased site understanding. Prior to 2014, the Canadian design codes specified a single resistance factor for each limit state. In other words, these codes made no allowance for changes in the resistance factor as changes in the level of site understanding and, for that matter, of failure consequences, occur.

The overall target safety level of any design should depend on at least three factors: (1) the uncertainty in the loads, (2) the uncertainty in the resistance, and (3) the severity of the failure consequences. In most modern codes, these three items are assumed independent of one another and are thus treated separately. The load factors handle the uncertainties in the loads and, on the load side, failure consequences are handled by applying an importance factor to the more uncertain and site specific loads (e.g. earthquake, snow, and wind). In North America, the importance factor multiplies the load factor to adjust the load exceedance probability (or return period) and thus the target reliability index. In Europe, the K_{FI} factor, defined in Annex B of *Eurocode 1990 – Basis of Structural Design* [2] is similar in nature. Uncertainties in resistance are handled by resistance factors that are usually specific to the material used in the design (e.g. φ_c for concrete, φ_s for steel, etc.). Note that in North America and within this paper, these factors are applied in a multiplicative fashion, whereas in Europe they are applied in a divisive fashion.

When dealing with a highly variable and site specific material such as the ground, it makes sense to apply a resistance factor that depends on both the resistance uncertainty and on the consequences of failure. The basic idea is that the overall factor applied to the geotechnical resistance should vary with both uncertainty and failure consequence. Increased site investigation should lead to lower uncertainty and a higher resistance factor, and thus a more economical design. Similarly, for geotechnical systems with high failure consequences, e.g. failure of the foundation of a major multi-lane highway bridge in a large city, the overall resistance factor should be decreased to provide a decreased maximum

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Nomenclature

CPT	cone penetration test for in-situ soil testing
LRFD	Load and Resistance Factor Design
LSD	Limit States Design
SLS	Serviceability Limit State
ULS	Ultimate Limit State
a_i	pile settlement prediction parameters, $i = 0, 1, 2$
B	width of shallow foundation
B_i	iterative guesses at shallow foundation width
c	cohesion (random)
c_g	equivalent cohesion which gives same resistance as spatially variable cohesion (random)
$\hat{c}$	characteristic cohesion, (geometric) average of sample values (random)
C	edge length of averaging domain, V_f, under a shallow foundation
d	pile diameter
D	depth of soil sample
$\hat{E}$	characteristic elastic modulus, (geometric) average of sample values (random)
E_g	equivalent uniform soil elastic modulus which gives same response as actual spatially variable elastic modulus (random)
E_{med}	median elastic modulus
F_D	true dead load (random)
F_L	true live load (random)
F_T	true total load (random)
$\hat{F}_D$	characteristic dead load
$\hat{F}_{iu}$	i'th characteristic load effect at ULS
$\hat{F}_{is}$	i'th characteristic load effect at SLS
$\hat{F}_L$	characteristic live load
$\hat{F}_T$	characteristic total load
$\hat{F}_{T_u}$	total design load at ULS
$\hat{F}_{T_s}$	total design load at SLS
H	pile length or depth to bedrock
I_{iu}	importance factor corresponding to i'th characteristic load effect at ULS
I_{is}	importance factor corresponding to i'th characteristic load effect at SLS
I_p	settlement influence factor
K_{FI}	Eurocode reliability differentiation factor
$\hat{N}_c$	characteristic bearing capacity factor (base on characteristic soil properties)
N_{c_g}	equivalent bearing capacity factor which gives same response as actual spatially variable ground
p	pile perimeter length
p_f	probability of failure
p_m	lifetime maximum acceptable failure probability
$\mathbf{P}[\cdot]$	probability operator
r	distance between soil sample and foundation centerlines
R	ground resistance
$\hat{R}_s$	characteristic serviceability ground resistance based on characteristic soil properties
$\hat{R}_u$	characteristic ultimate ground resistance based on characteristic soil properties
s	width parameter of bounded tanh distribution
u_1	settlement influence factor
v_E	coefficient of variation of elastic modulus
v_c	coefficient of variation of cohesion
v_L	coefficient of variation of live load
v_D	coefficient of variation of dead load
v_{F_T}	coefficient of variation of total load
V_f	averaging volume of the ground under or around the foundation
V_s	averaging volume of the soil sample
V_{fs}	average correlation coefficient between the sample volume and the averaging volume under the foundation
W	true total load times ratio of characteristic to equivalent ground properties (random)
$\underset{\sim}{x}$	vector containing spatial position
$\tilde{\alpha}$	adhesion coefficient
α_{iu}	load factor corresponding to the i'th load effect at ULS
α_{is}	load factor corresponding to the i'th load effect at SLS
α_D	dead load factor
α_L	live load factor
β	reliability index
Δx	width of soil sample
δ	true foundation settlement (random)
δ_{max}	maximum acceptable foundation settlement
$\underset{\sim}{\eta}$	vector containing spatial position
γ_f	variance reduction factor due to averaging over domain V_f
γ_s	variance reduction factor due to averaging over domain V_s
γ_{fs}	average correlation coefficient between the ground properties within domains V_f and V_s
μ_c	mean cohesion
μ_D	mean dead load
μ_E	mean elastic modulus
μ_{F_T}	mean total load
μ_L	mean live load
$\mu_{\ln F_T}$	mean of the logarithm of total load
$\mu_{\ln W}$	mean of the logarithm of W
$\mu_{\ln c}$	mean of the logarithm of cohesion
$\mu_{\ln c_g}$	mean of the logarithm of the cohesion (geometrically) averaged over domain V_f
$\mu_{\ln \hat{c}}$	mean of the logarithm of characteristic cohesion
$\mu_{\ln N_c}$	mean of the logarithm of bearing capacity factor
$\mu_{\ln N_{c_g}}$	mean of the logarithm of the bearing capacity factor (geometrically) averaged over domain V_f
$\mu_{\ln \hat{N}_c}$	mean of the logarithm of the characteristic bearing capacity factor
$\hat{\mu}_B$	estimated mean footing width
μ_ϕ	mean friction angle
μ_{N_c}	mean bearing capacity factor
ρ	correlation coefficient between the ground properties (transformed to Gaussian space) at two points
θ	random field correlation length
$\sigma_{\ln F_T}$	standard deviation of logarithm of total load
σ_{F_T}	standard deviation of total load
$\sigma_{\ln W}$	standard deviation of the logarithm of W
$\sigma_{\ln R}$	standard deviation of the logarithm of ground resistance
φ_c	resistance factor for concrete
φ_g	geotechnical resistance factor at either ULS or SLS
φ_{gu}	geotechnical resistance factor at ULS
φ_{gs}	geotechnical resistance factor at SLS
φ_s	resistance factor for steel
φ_o	moderate or average resistance factor
$\hat{\phi}$	characteristic soil friction angle
ϕ_{min}	minimum soil friction angle
ϕ_{max}	maximum soil friction angle
Ψ	consequence factor at either ULS or SLS
Ψ_u	consequence factor at ULS
Ψ_s	consequence factor at SLS
$\underset{\sim}{\xi}$	vector containing spatial position

acceptable failure probability. Similar to the multiplicative approach taken in structural engineering, where the overall load factor is a product of a load factor and an importance factor, the overall resistance factor applied to geotechnical resistance is taken here to consist of two parts which are multiplied together:

(1) a resistance factor, φ_{gu} or φ_{gs}, which accounts for resistance uncertainty. This factor basically aims to achieve a target maximum acceptable failure probability equal to that used for geotechnical designs for typical failure consequences, e.g. lifetime failure probability of approximately 1/5000 for ultimate limit states or 1/500 for serviceability limit states. The subscript g refers to 'geotechnical' (or 'ground'), while the subscripts u and s refer to ultimate and serviceability limit states, respectively.
(2) a consequence factor, Ψ_u or Ψ_s, which accounts for failure consequences at either the ultimate or serviceability limit states, respectively. The consequence factor will be greater than 1.0 if failure consequences are low and less than 1.0 if failure consequence exceed those of typical geotechnical systems. The basic idea of the consequence factor is to adjust the maximum acceptable failure probability of the design down (higher reliability) for high failure consequences, or up (lower reliability) for low failure consequences.

This paper will consider the limit state design (LSD) of both shallow and deep foundations within a load and resistance factor design (LRFD) framework. The goal is to provide a single theoretical model which can be used to determine the resistance and consequence factors required to achieve a target maximum acceptable failure probability for a variety of geotechnical design problems. The methodology and results presented here are intended for use in the total resistance factor approach, which is Design Approach 2 in the Eurocodes, and generally used in North America.

Within the LRFD framework, geotechnical designs proceed by adjusting the resistance parameters (usually the foundation geometry) so that the factored geotechnical resistance is at least equal to the effect of factored loads. For example, for ultimate limit states (ULS), this means that the geotechnical design should satisfy an equation of the form

$$\Psi_u \varphi_{gu} \hat{R}_u \geqslant \sum I_{iu} \alpha_{iu} \hat{F}_{iu} \tag{1}$$

in which Ψ_u is a consequence factor, φ_{gu} is the geotechnical resistance factor, and $\hat{R}_u$ is the characteristic ultimate resistance, all at the ULS. In Canada, the characteristic resistance is assumed to be the best estimate of the ground resistance using ground properties which are (cautious) estimates of the mean ground properties yielding a (cautious) estimate of the mean ground resistance. The right-hand-side consists of I_{iu}, an importance factor, multiplying the ith factored load effect, $\alpha_{iu}\hat{F}_{iu}$. A similar equation must be satisfied for serviceability limit states (SLS), with the subscript u replaced by s, i.e.,

$$\Psi_s \varphi_{gs} \hat{R}_s \geqslant \sum I_{is} \alpha_{is} \hat{F}_{is} \tag{2}$$

As mentioned previously, the load factors, α_{iu} or α_{is}, typically account for uncertainty in loads, and are greater than 1.0 for ultimate limit states but often assumed equal to 1.0 for serviceability limit states. The geotechnical resistance factor, φ_{gu} or φ_{gs}, is typically less than 1.0 and accounts for uncertainties in the geotechnical parameters and models used to estimate the characteristic geotechnical resistance, $\hat{R}_u$ or $\hat{R}_s$, along with other sources of error (e.g. model error). The consequence factor, Ψ_u or Ψ_s, and the importance factor, I_{iu} or I_{is}, are employed to adjust the target reliability level to account for different magnitudes of failure consequences. As discussed earlier, the importance factor appears on the load side of Eqs. (1) and (2) in order to account for failure consequences and is generally applied to site specific and highly uncertain load distributions (usually snow, wind, and earthquake). Because the ground is also site specific and highly uncertain, it makes sense to apply a similar *consequence* factor to the resistance side of Eqs. (1) and (2) and so adjust the factored resistance to account for failure consequences, particularly in those cases not covered by the load side importance factor. Further research needs to be performed to establish the interaction between the importance and consequence factors and their combined effect on failure probability. For example, for frictional soils, changing the total load factor may have little effect on the required resistance design and it could be that the consequence factor needs to be applied simultaneously with the importance factor in this case.

Since the focus of this work is on developing a unified theory to predict resistance and consequence factors, applied to the characteristic resistance, the importance factors, I_{iu} and I_{is}, will be assumed to have values 1.0. The interaction between the importance and consequence factors is beyond the scope of this paper. In addition, only dead (permanent) and live (variable) loads will be considered in this study. If the total design loads, $\hat{F}_{T_u}$ and $\hat{F}_{T_s}$, for ULS and SLS respectively, are defined as the sum of the factored characteristic loads,

$$\hat{F}_{T_u} = \alpha_L \hat{F}_L + \alpha_D \hat{F}_D \tag{3a}$$

$$\hat{F}_{T_s} = \hat{F}_L + \hat{F}_D \tag{3b}$$

where it is assumed that the SLS load factors are 1.0, then the LRFD Eqs. (1) and (2) simplify to

$$\Psi_u \varphi_{gu} \hat{R}_u \geqslant \hat{F}_{T_u} \tag{4a}$$

$$\Psi_s \varphi_{gs} \hat{R}_s \geqslant \hat{F}_{T_s} \tag{4b}$$

Three failure consequence levels will be considered in this paper;

(1) *high consequence:* failure of the supported structure has large safety and/or financial consequences (e.g., hospitals, schools, and lifeline highway bridges),
(2) *typical consequence:* has failure consequences typical of the majority of civil engineering projects, and
(3) *low consequence:* failure of the supported structure has little or no safety and/or financial consequences (e.g., low use storage facilities or low use bridges).

Most designs will be aimed at the typical failure consequence level. The target maximum acceptable lifetime failure probabilities, p_m, assumed in this study are as shown in Table 1, along with their corresponding reliability indices, β, (shown in brackets). The values shown for ULS approximately span the reliability index range suggested in Canadian structural design codes. For example, the Canadian Highway Bridge Design Code [3] specifies an annual target reliability index of 3.75, which corresponds to a 50-year lifetime target reliability index of somewhere between 2.7 and 3.5, depending on the assumptions made about inter-year dependencies. The

Table 1
Targeted theoretical maximum lifetime failure probabilities, p_m, and equivalent reliability indices, β (shown parenthesized) for ULS and SLS foundation design.

Consequence level	ULS	SLS
High	1/10,000 (3.7)	1/1000 (3.1)
Typical	1/5000 (3.5)	1/500 (2.9)
Low	1/1000 (3.1)	1/100 (2.3)

values shown in Table 1 may thus be somewhat on the high side for Canada, but may be on the low side with respect to the Eurocode [2] which suggests a higher consequence reliability index for a 50-year lifetime of 4.3. What the societally acceptable target reliability indices should be is evidently still a topic worthy of continued world-wide investigation. Note also that despite the differing target reliability indices between Canada and Europe, the design factors used in both regions result in pretty much the same overall factor of safety [9], suggesting that the achieved reliability in both jurisdictions are actually quite similar.

Note that the target failure probabilities shown in Table 1 assume some redundancy (as typically required in structural codes), so that the actual system lifetime failure probability is usually less than the component maximum lifetime failure probability, p_m. The effect of redundancy in geotechnical components on reliability, which may lead to adjustment of the resistance factors, is a topic currently under investigation by the authors.

2. Theoretical failure probability and derived design factors

The theoretical framework required to estimate the failure probability of a geotechnical system should consider;

(1) uncertainty in the loads, including consideration of the distribution of static (dead) loads and the extreme value distribution of dynamic (live) loads over the lifetime of the geotechnical system and its supported structure, and
(2) uncertainty in the resistance, including random field models of the ground to characterize its natural spatial variability, along with prediction model uncertainty, and uncertainty in the ground strength parameters (due to measurement errors and lack of sufficient sampling) within the zone of influence under and around the foundation being designed. In this paper, only spatial variability is considered, not uncertainty in the prediction model nor in the measurement of the ground strength parameters. This means that the resistance factors derived here are *upper bounds* in the event that the ground parameter variability and correlation length are known precisely. Otherwise, the assumption of a worst case correlation length (to be discussed later) or a slightly higher coefficient of variation can be used to account for the model and measurement errors which are not directly accounted for here (see, e.g. [12], for a more detailed discussion of the effects of measurement errors). Alternatively, designers with an increased understanding of the site's geology, experience with similar sites, etc., will be able to approach the upper bounds presented here.

In its simplest form, a geotechnical system fails if its resistance, R, is less than the supported total load, F_T, any time during the system's design life. The system resistance R is often a complex function of the actual ground properties over space and time – usually corresponding to some sort of minimum over time of all minimum

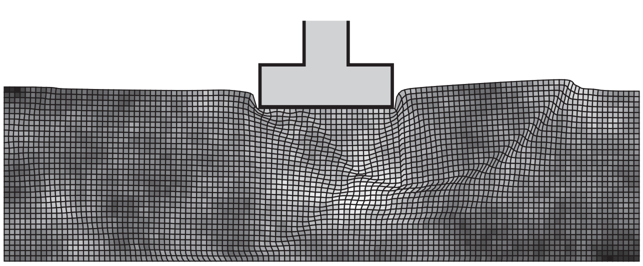

Fig. 1. Bearing failure of a shallow foundation on a spatially variable soil.

'strength' failure mechanisms through space. For example, bearing failure of a shallow foundation occurs when the weakest path through the ground has insufficient shear strength to resist the applied load. Fig. 1 illustrates a bearing failure mechanism which might occur at an instant in time during the design life of a footing. Rather than the traditionally assumed symmetric double log-spiral failure mechanism predicted when the ground properties are spatially constant, the failure mechanism that occurs when ground properties vary spatially follows the weakest path, resulting in non-symmetric and sometimes quite erratic failure paths. In particular, in Fig. 1, the weaker (whiter) ground to the right of the footing attracts the failure mechanism. The 'weakest path' failure mechanism is on average weaker than the uniform symmetric failure mechanism suggested by traditional analysis (based on the mean strength), which implies that traditional models are unconservative when based on the mean. Fig. 1 is taken from a previous random finite element method (RFEM) study which was one of several used to validate the theory presented in this paper [8]. The results presented here do not, however, make direct use of the finite element method.

The major challenge in reliability-based design is how to capture the weakest path behavior of the ground in a way that is simple enough to use in practice. The key to answering this question is to replace the spatial variability of the ground by a single 'equivalent' random variable which yields the same probabilistic behavior as the actual spatially variable ground. Since this paper is also concerned with uncertainty in the estimation of ground properties, two random variables are used for each ground property in the proposed unified model. The first is an equivalent single random variable which yields the same foundation response as does the actual random field (e.g. c_g, to be defined shortly). The second is a *characteristic* random variable which is the sample *estimate* of the equivalent random variable (e.g., $\hat{c}$, also to be defined shortly). For example, if a realization of an actual spatially variable $c-\phi$ field provides a shallow foundation bearing capacity of 1000 kN, then the *equivalent* single values of c and ϕ would be those which provide a bearing capacity of 1000 kN. A sample of the spatially variable $c-\phi$ field would allow 'characteristic' estimates of these 'equivalent' values. If the sample is accurate, then the characteristic values will be close to the equivalent values and the resulting foundation design should have a lower failure probability.

Consider, for example, the settlement of a shallow foundation where performance failure is defined as the event that the actual foundation settlement, δ, exceeds the serviceability limit, δ_{max}, i.e.,

$$p_f = \mathbf{P}[\delta > \delta_{max}] \tag{5}$$

The actual settlement, δ, is a function of the random loads the foundation sustains over time, the foundation geometry, and the random (usually non-linear) compressibility field of the ground under the footing. Thus, δ is a very complicated function of many random variables. Nevertheless, δ is a single random variable which has some distribution. If that distribution can be found, then p_f can be determined.

To illustrate the process in a geotechnical context, consider the bearing failure of a strip footing supported by a $c-\phi$ soil, as shown in Fig. 1 (following [8]). To simplify the illustration, the soil will be considered weightless with no foundation embedment nor surcharge. It is noted that the lack of weight, embedment, and surcharge only affects the required resistance factors if these additional design parameters add significantly to the overall *uncertainty* of the mean resistance prediction, in which case a lower resistance factor should be selected.

In this example, the characteristic resistance is

$$\hat{R}_u = B\hat{c}\hat{N}_c \tag{6}$$

where B is the footing width, $\hat{c}$ is the characteristic cohesion, and the characteristic bearing capacity factor, $\hat{N}_c$, is given by (see e.g., [18,14,13], the form shown here was developed by Griffiths et al. [10])

$$\hat{N}_c = \frac{\exp\{\pi \tan\hat{\phi}\}\left(\tan\hat{\phi} + \sqrt{1+\tan^2\hat{\phi}}\right)^2 - 1}{\tan\hat{\phi}} \tag{7}$$

Since this example is of a strip footing, the units of $\hat{R}_u$ are force per unit length along the strip footing.

The characteristic ground parameters (e.g., cohesion and friction angle) are obtained through a site exploration program. Although the definition of 'characteristic' varies quite widely around the world, it is assumed here that the characteristic values are 'a cautious estimate of the mean ground parameter'. In North America, the characteristic values are usually based on some sort of conservative estimate of the mean (see, e.g., [1,19]), which is generally somewhat below the (estimated) mean. In this paper, the characteristic values are defined as geometric averages for shallow foundations (which are always somewhat below the mean being low strength dominated) and as arithmetic averages for deep foundations (which are best estimates of the mean). See Fellin and Oberguggenberger [5] for more rigorous definitions of ground shear strength parameters which include, for example, the effect of correlation between cohesion and friction angle. For simplicity, cohesion and friction angle are assumed independent in this paper.

Using Eq. (6) in Eq. (7), the LRFD equation becomes

$$\Psi_u \varphi_{gu} B\hat{c}\hat{N}_c \geqslant \alpha_L \hat{F}_L + \alpha_D \hat{F}_D \tag{8}$$

which, taken at the equality, allows the footing to be designed,

$$B = \frac{\alpha_L \hat{F}_L + \alpha_D \hat{F}_D}{\Psi_u \varphi_{gu} \hat{c}\hat{N}_c} = \frac{\hat{F}_{T_u}}{\Psi_u \varphi_{gu} \hat{c}\hat{N}_c} \tag{9}$$

Failure of the footing occurs if the actual total load on the footing, $F_T = F_L + F_D$, where F_L is the actual live (variable) load and F_D is the actual dead (permanent) load (both random), exceeds the actual (random) resistance. The probability of failure is thus

$$p_f = \mathbf{P}[F_T > c_g N_{c_g} B] \tag{10}$$

where c_g and N_{c_g} are some sort of averages of the random cohesion and friction fields, taken in the vicinity of the footing, such that the product $c_g N_{c_g} B$ has the same distribution as the actual resistance of the spatially variable ground. Past research by the authors (see [7]) has shown that c_g and N_{c_g} are well approximated by suitably selected geometric averages of c and ϕ in the vicinity of the foundation. The appropriate averaging regions are suggested in the following sections.

Substituting Eq. (9) into Eq. (10) and collecting all random variables to the left side of the inequality leads to

$$p_f = \mathbf{P}\left[F_T \frac{\hat{c}\hat{N}_c}{c_g N_{c_g}} > \frac{\hat{F}_{T_u}}{\Psi_u \varphi_{gu}}\right] \tag{11}$$

If we let

$$W = F_T \frac{\hat{c}\hat{N}_c}{c_g N_{c_g}} \tag{12}$$

then the failure probability can be written in general terms (for either ULS or SLS by dropping the u subscript on the resistance and consequence factors) as

$$p_f = \mathbf{P}\left[W > \frac{\hat{F}_T}{\Psi \varphi_g}\right] \tag{13}$$

The random variables on the right-hand-side of Eq. (12) are all assumed to be lognormally distributed. This is often a reasonable (and conservative) assumption, especially for those variables which are defined as geometric averages, since geometric averages tend to a lognormal distribution by the central limit theorem. If this assumption is true, then W is also (at least approximately) lognormally distributed, so that

$$p_f = \mathbf{P}[\ln W > \ln \hat{F}_T - \ln \Psi\varphi_g] = 1 - \Phi\left(\frac{\ln \hat{F}_T - \ln\left(\Psi\varphi_g\right) - \mu_{\ln W}}{\sigma_{\ln W}}\right) \tag{14}$$

where Φ is the cumulative standard normal distribution function. Noting that the probability of failure can be expressed in terms of the reliability index, β, as $p_f = 1 - \Phi(\beta)$, then an explicit expression for the total factor applied to the resistance, is

$$\Psi\varphi_g = \frac{\hat{F}_T}{\exp\{\mu_{\ln W} + \beta\sigma_{\ln W}\}} \tag{15}$$

where, for design, the reliability index is taken as the target specified as in Table 1; $\beta = \Phi^{-1}(1 - p_m)$.

Although the definition of W changes slightly from problem to problem, once it is defined, Eqs. (13)–(15) can be used to determine the failure probability and total resistance factor for all four geotechnical problems considered in this paper, so long as suitable averaging regions can be found under or around the geotechnical system. It is expected that these equations can also be used for most other geotechnical problems, with the possible exception of slope stability and problems where the soil acts as both the load and the resistance (e.g., some retaining walls). In addition, it will be shown that usually $\mu_{\ln W} = \mu_{\ln F_T}$ and that $\sigma_{\ln W}$ has a form which is common to most geotechnical problems.

For the bearing capacity of a strip footing currently under consideration, the parameters of the lognormally distributed random variable W are obtained by looking at the mean and variance of $\ln W$, where

$$\ln W = \ln F_T + \ln \hat{c} - \ln c_g + \ln \hat{N}_c - \ln N_{c_g} \tag{16}$$

Now assume that $\hat{c}$ and c_g are defined as geometric averages over the sample volume and over some suitable volume under/around the foundation, respectively. If so, then $\ln\hat{c}$ and $\ln c_g$ are arithmetic averages of $\ln c(\underset{\sim}{x})$ over the same volumes,

$$\ln \hat{c} = \frac{1}{V_s}\int_{V_s} \ln c(\underset{\sim}{x})\, d\underset{\sim}{x} \tag{17a}$$

$$\ln c_g = \frac{1}{V_f}\int_{V_f} \ln c(\underset{\sim}{x})\, d\underset{\sim}{x} \tag{17b}$$

where $\underset{\sim}{x}$ is spatial position, V_s is the volume of the soil sample (assumed to be contiguous for simplicity – otherwise Eq. (17a) a becomes a discrete sum), and V_f is a suitable volume of the averaging region in the vicinity of the foundation. The main difficulty with the solution of Eqs. (14) and (15) is with the selection of an appropriate averaging region, V_f.

In order to solve Eqs. (14) and (15), the mean and variance of $\ln W$ must be found. The mean is relatively simple if the ground is assumed to be statistically stationary (the mean and covariance structure remains constant over space), so that

$$\mu_{\ln\hat{c}} = \mu_{\ln c_g} = \mu_{\ln c} \tag{18a}$$

$$\mu_{\ln\hat{N}_c} = \mu_{\ln N_{c_g}} = \mu_{\ln N_c} \tag{18b}$$

which gives

$$\mu_{\ln W} = \mu_{\ln F_T} + \mu_{\ln\hat{c}} - \mu_{\ln c_g} + \mu_{\ln\hat{N}_c} - \mu_{\ln N_{c_g}} = \mu_{\ln F_T} \tag{19}$$

The cancellation of all of the mean ground parameters from $\mu_{\ln W}$ is typical if the random field is stationary. Note that if the random field is not stationary in the mean, the characteristic resistance (Eq. (6)) need only be separated into two parts; a deterministic part which is determined from the mean trend, and a residual part which is stationary and follows the above procedure. The results presented above would then still apply once the mean trend has been removed from the random field and used directly in the prediction of the characteristic geotechnical resistance.

The variance of $\ln W$ is complicated by the random field model of the ground. As mentioned previously, the basic idea is to replace the spatial variability of the actual ground with suitably defined local averages. Fig. 2 illustrates the local averages involved: one local average under the footing is within the region V_f, and if the size of V_f is properly selected, then the ground properties averaged over V_f will have the same bearing capacity distribution as the actual ground. Because the bearing failure follows the weakest path through the ground, a geometric average has been found to be appropriate [8]. Similarly, in order to perform the design, the ground will have been sampled at some location and then the characteristic ground parameters used in the design would be some sort of average of the sample values. If it is assumed that the soil sample is actually a CPT sounding of depth D at some location r away from the center of the footing, then the characteristic ground parameters would be an average of the observations over the volume V_s. It will be assumed here that a CPT sounding is reflecting the soil's strength parameters over a region around the cone of width Δx and it is further assumed that the appropriate average to use is again a geometric average.

If the load, F_T, and ground strength parameters, in this case c and ϕ, are assumed to be mutually independent then, to at least first order (see [8], for details),

$$\sigma^2_{\ln W} = \sigma^2_{\ln F_T} + \left(\sigma^2_{\ln c} + \sigma^2_{\ln N_c}\right)\left[\gamma_f + \gamma_s - 2\gamma_{fs}\right] \tag{20}$$

where γ_f is the variance reduction factor due to (geometric) averaging over a suitable region (V_f) under or around the foundation, γ_s is the variance reduction factor due to (geometric) averaging of the soil sample (within V_s), and γ_{fs} is the average correlation coefficient between the region V_f and the region V_s. The last is really a reflection of how well the soil sample estimates the nature of the ground under the footing. As r increases, it is expected that γ_{fs} will decrease, indicating that the ground conditions at the footing are less well predicted by the sample. In this way, the degree of 'site understanding' can be partially reflected by adjusting r. If a designer has high confidence in their understanding of the ground parameters under the footing being designed, then that corresponds to a small value of r in this model. Conversely, low understanding of ground properties under the footing corresponds to a large value of r. In detail,

$$\gamma_f = \frac{1}{V_f^2}\int_{V_f}\int_{V_f}\rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\,d\underset{\sim}{\eta}\,d\underset{\sim}{\xi} \tag{21a}$$

$$\gamma_s = \frac{1}{V_s^2}\int_{V_s}\int_{V_s}\rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\,d\underset{\sim}{\eta}\,d\underset{\sim}{\xi} \tag{21b}$$

$$\gamma_{fs} = \frac{1}{V_f V_s}\int_{V_f}\int_{V_s}\rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})\,d\underset{\sim}{\eta}\,d\underset{\sim}{\xi} \tag{21c}$$

where $\underset{\sim}{\eta}$ and $\underset{\sim}{\xi}$ are spatial positions and ρ returns the correlation coefficient between two points in the ground separated by distance $\underset{\sim}{\eta} - \underset{\sim}{\xi}$. In all of the examples considered here, ρ gives the correlation between the ground properties *transformed into Gaussian space*. For example, if c is assumed lognormally, as it is, then $\rho(\underset{\sim}{\eta} - \underset{\sim}{\xi})$ gives the correlation coefficient between $\ln c$ at two points in the ground separated by distance $\underset{\sim}{\eta} - \underset{\sim}{\xi}$. Eq. (21) can be evaluated using Gauss Quadrature or a similar numerical integrator. In this work, the correlation coefficient is assumed to be Markovian in nature,

$$\rho(\tau) = \exp\left\{\frac{-2|\tau|}{\theta}\right\} \tag{22}$$

where θ is the correlation length (see, e.g., [7]).

The averaging volume of the sample, V_s, is usually at least approximately known and will be one of the following values in this study;

(1) for 1-D averaging, $V_s = D$,
(2) for 2-D averaging, $V_s = \Delta x \times D$,
(3) for 3-D averaging, $V_s = \Delta x \times \Delta x \times D$.

The main challenge at this point is to decide on the appropriate size of the averaging volume, V_f. The geotechnical failure mechanism below (or around) the foundation usually involves some averaging of the strength or deformation properties of the ground and the size of V_f should properly reflect the actual averaging. This means that V_f is dependent on the size of the foundation itself, which means that, strictly speaking, V_f is not known until after the foundation is designed (which means that the resistance factors need to be known before V_f can be determined).

In some cases, the variance reduction factor, γ_f, and the average correlation coefficient, γ_{fs}, are not very sensitive to fairly significant changes in V_f. This means that V_f can sometimes be reasonably approximated by using a 'typical' design, perhaps based on the mean ground properties and a typical (or traditional) resistance factor. In other cases, the variances and correlations are more sensitive to the size of V_f, in which case an iterative approach provides better results. If iteration is required, the basic algorithm to be used is as follows;

(1) choose a reasonable starting value for the total resistance factor ($\Psi\varphi_g$),
(2) find the minimum foundation dimensions which satisfy the LRFD requirements (see Eq. (4)),
(3) set the V_f averaging domain as some appropriate function of the foundation dimensions (this step will be discussed in more detail for each geotechnical problem considered shortly),
(4) compute γ_f, γ_s, and γ_{fs} according to Eq. (21),
(5) use Eq. (20) to compute $\sigma^2_{\ln W}$,

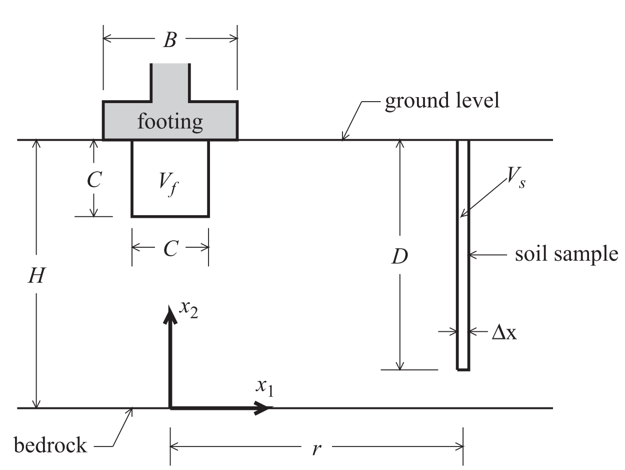

Fig. 2. Averaging regions used to predict probability of bearing capacity failure.

(6) update the total resistance factor $(\Psi\varphi_g)$ according to Eq. (15). Also, compute the failure probability, p_f, according to Eq. (14) if desired. If the total resistance factor has changed by only within some relative error tolerance (e.g., 0.001), or if p_f is within some relative error tolerance from the target p_m, then the iterations can stop. Otherwise, repeat from step 2 using the adjusted value of the total resistance factor.

Once the total resistance factor, $(\Psi\varphi_g)$, has been determined for a variety of values of the target failure probability, p_m (see Table 1), the consequence factor, Ψ, is determined rather simply. Consider again the bearing capacity problem and assume that the total resistance factor has been determined for $p_m = 1/1000$ (low consequence), $p_m = 1/5000$ (typical consequence) and $p_m = 1/10,000$ (high consequence). Denoting the corresponding total resistance factors $(\Psi_u\varphi_{gu})_{low}$, $(\Psi_u\varphi_{gu})_{typ}$, and $(\Psi_u\varphi_{gu})_{high}$, then assuming that $\Psi_u = 1.0$ for the typical case, we get

$$\text{low consequence}: \ \Psi_u = \frac{(\Psi_u\varphi_{gu})_{low}}{(\Psi_u\varphi_{gu})_{typ}} \tag{23a}$$

$$\text{high consequence}: \ \Psi_u = \frac{(\Psi_u\varphi_{gu})_{high}}{(\Psi_u\varphi_{gu})_{typ}} \tag{23b}$$

3. Factors for the ULS design of shallow foundations

The theory required to estimate the failure probability, and thus the required resistance and consequence factors for the ULS bearing capacity design of a shallow foundation, was presented in the previous section. A specific case, with parameters as given in Table 2, will be considered to illustrate the results. Note that the mean values appearing in Table 2 are not important since $\Psi_u\varphi_{gu}$ depends primarily on the variance and spatial variability (correlation length, θ) of the ground. This means that the results presented here are generally applicable for the typical levels of uncertainty in the ground and load parameters assumed in Table 2, i.e., $v_c = 0.3, v_\phi = 0.2$, and live and dead load coefficients of variation of 0.3 and 0.15, respectively. The total design load, $\hat{F}_{T_u}$, assumes live and dead load factors of $\alpha_L = 1.5$ and $\alpha_D = 1.25$ along with live and dead load bias factors of 1.41 and 1.18, respectively. The load bias factors are the ratio of the characteristic loads to the mean loads. The values used here are as assumed in the study by Fenton et al. [8]. This gives a total design load of $\hat{F}_T = 1.5(1.41)(200) + 1.25(1.18)(600) = 1308$ kN/m, as is also shown in Table 2.

The main features of the ULS reliability-based design of a shallow foundation can be found in Fenton et al. [8]. They found that in a 2-D analysis, V_f is well approximated by a square of dimension $C \times C$ centered under the footing (see Fig. 2), where C is about 80% of the mean depth of the classical wedge failure zone given by Prandtl,

Table 2
Parameters used in the investigation of required resistance and consequence factors for the ULS design of shallow foundations.

Parameter	Value
$\mu_c,\ v_c$	100 kN/m, 0.3
$\mu_\phi,\ v_\phi$	$20^\circ,\ 0.2$
$\mu_L,\ v_L$	200 kN/m, 0.3
$\mu_D,\ v_D$	600 kN/m, 0.15
$\hat{F}_{T_u}$	1308 kN/m
$\Delta x,\ H$	0.15 m, 4.8 m
θ	0.1–50 m

$$C = \frac{0.8}{2}\hat{\mu}_B \tan\left(\frac{\pi}{4} + \frac{\mu_\phi}{2}\right) \tag{24}$$

In the above, $\hat{\mu}_B$ is an estimate of the mean footing width obtained by evaluating Eq. (9) at the mean of the ground properties with a reasonable estimate of the average $\Psi_u\varphi_{gu}$ equal to about 0.7,

$$\hat{\mu}_B = \frac{\hat{F}_{T_u}}{0.7\mu_c\mu_{N_c}} \tag{25}$$

Using this result in Eq. (24) to define $V_f = C \times C$ allows the results of the previous section to be used to find the failure probability (Eq. (14)) and total resistance factor (Eq. (15)) required to achieve a target failure probability, p_m.

For the uncertainty levels given in Table 2, the resistance factors required to achieve a typical lifetime maximum acceptable failure probability of $p_m = 1/5000$ are shown in Fig. 3.

In Fig. 3, the correlation length, θ, is varied from a low of 0.1 m to a high of 50 m. Low values of θ lead to soil properties varying rapidly spatially, while high values mean that the soil properties vary only slowly with position. A large correlation length, of say $\theta = 50$ m, means that soil samples taken well within 50 m from the footing location (e.g. at $r = 10$ m) will well predict the soil properties under the footing so that lower failure probabilities are expected. In turn, this means that for fixed target failure probability, the required resistance factor increases towards 1.0 as the correlation length increases.

Interestingly, at the other extreme when the correlation length becomes very small, the required resistance factor is seen again to increase. The reason for this is that the characteristic value derived from the soil sample is generally some form of average – and the geometric average is used here. When θ is smaller than the averaging volume, the average itself tends towards the median, and becomes equal to the median (with no variability) when $\theta = 0$. Since both the sample and the failure mechanism under the footing involve geometric averaging, then as $\theta \to 0$, both the sample average and the average of the ground under the footing become the same (equal to the median). In this case, the sample will again accurately reflect the conditions under the footing, leading to reduced failure probabilities, or, equivalently, increased required resistance factors.

It is somewhere in between these two extremes that the failure probability reaches a maximum. It turns out that it is when the correlation length is approximately equal to the distance between the sample and the footing that the sample gives the poorest prediction of the conditions under the footing and the highest failure probability or lowest resistance factor. This 'worst case' correlation length can be seen in Fig. 3 to be around 2 m when $r = 0$ m, around 5 m when $r = 5$ m, and around 10 m when $r = 10$ m.

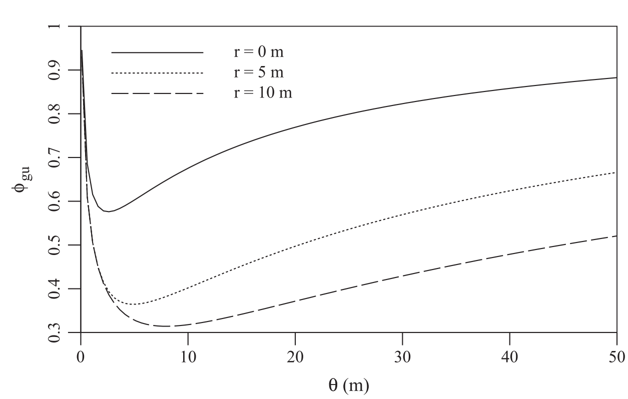

Fig. 3. Resistance factors required to achieve $p_m = 1/5000$ ($\beta = 3.5$) for the ULS design of a shallow foundation (using coefficients of variation specified in Table 2).

The authors note that the correlation length is fundamentally difficult to estimate, since it is highly biased in the presence of correlation (which is what it is trying to estimate). In other words, the correlation length will often not be known in practice. The presence of a "worst case" correlation length is actually very valuable for design since it relieves us of the requirement to actually determine the true correlation length. The "worst case" correlation length can be directly used to yield conservative designs.

It can also be seen from Fig. 3, that if an intermediate level of site understanding is assumed (corresponding to $r = 5$ m), then the worst case correlation length estimate of the required resistance factor is about $\phi_{gu} = 0.4$. A typical resistance factor for bearing capacity appearing in North American codes is about 0.5, so it could be that Fig. 3 is erring on the conservative side. However, the main value of probabilistic analyses such as presented here is to allow the determination of relative changes in target resistance factors as site understanding changes. For example, if the currently accepted resistance factor for the bearing capacity design of a foundation is 0.5, and it is assumed that typical site understanding corresponds to $r = 5$ m, then Fig. 3 suggests that if site understanding is reduced to $r = 10$ m, then the required resistance factor should be scaled down from 0.5 by the amount the $r = 5$ curve reduces to the $r = 10$ curve at the worst case, ie to about $0.5(0.31/0.37) = 0.42$.

Fig. 4 shows the low and high consequence factors, obtained using Eq. (23). Although the consequence factor is supposed to be primarily dependent on the target maximum acceptable failure probability appropriate for the failure consequence, there is some residual dependence on site understanding (r) and correlation length (θ). The dependence is slight, however, amounting to less than 5% relative change for high consequence (Fig. 4b) and less than 13% for low consequence (Fig. 4a). This dependence on r and θ is negligible compared to the changes seen in the resistance factor, which is supposed to depend on r and θ, (see φ_{gu} in Fig. 3) of up to 300%.

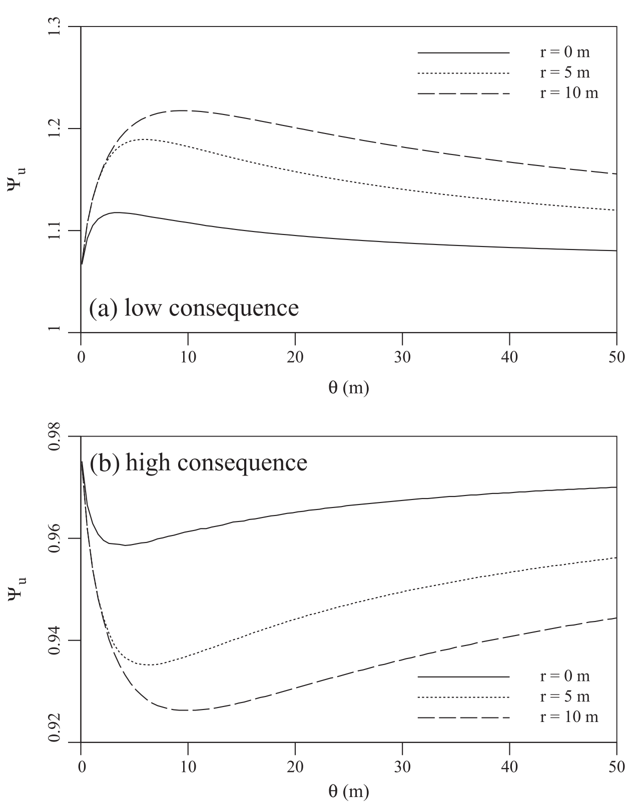

Fig. 4. ULS consequence factors for shallow foundations required to adjust $p_m = 1/5000$ ($\beta = 3.5$) to low consequence $p_m = 1/1000$ ($\beta = 3.1$) in (a) and to high consequence $p_m = 1/10,000$ ($\beta = 3.7$) in (b).

Noting that lower consequence factors result in lower failure probability, it can be seen that if Ψ_u is selected as 0.9 for high failure consequence cases, then the target maximum acceptable failure probability will be less than $p_m = 1/10,000$ for all cases of r and θ considered in Fig. 4b.

Since it is not so important to remain conservative when the failure consequences are already low, Fig. 4a suggests that $\Psi_u = 1.15$ might be appropriate for low failure consequence designs.

4. Factors for the SLS design of shallow foundations

The determination of resistance and consequence factors for the serviceability design of a shallow foundation involves using a spatially variable elastic modulus field representation of the ground to investigate failure probability. The details of this model can be found in Fenton et al. [6]. Note that while the use of an elastic model may not give the best estimate of the *mean* settlement of a foundation, it is the spatial variability of the ground that is important in the calibration of resistance and consequence factors. The random elastic field model provides an excellent vehicle to model spatial variability and its effect on the total resistance factor $\Psi_s \varphi_{gs}$.

The actual (random) settlement of a footing is thus predicted using Janbu's [11] elastic formula,

$$\delta = u_1 \frac{F_T}{BE_g} \tag{26}$$

where F_T is the total load on the footing (assumed vertical), B is the footing edge dimension (assumed square), E_g is the equivalent elastic modulus as 'seen' by the footing; i.e., the spatially uniform elastic modulus that gives the same settlement as the actual spatially variable elastic modulus. Using the random finite element method (RFEM), Fenton et al. [6] found that E_g was well approximated by a geometric average of the elastic modulus in a volume $V_f = B \times B \times H$, where H is the depth to bedrock (not to exceed about $2B$). The influence factor u_1 was determined by finite element analysis [6] to be

$$u_1 = 0.61\left(1 - e^{-1.18H/B}\right) \tag{27}$$

Eq. (26) can be used to determine the ground resistance corresponding to a given settlement, δ_{max}. Replacing F_T by the resistance R_s and letting $\delta = \delta_{max}$ in Eq. (26) gives

$$R_s = \frac{\delta_{max} B E_g}{u_1} \tag{28}$$

Replacing E_g with the estimate of the characteristic elastic modulus obtained from a soil sample, $\hat{E}$, yields the characteristic resistance

$$\hat{R}_s = \frac{\delta_{max} B \hat{E}}{u_1} \tag{29}$$

which, when substituted into the LRFD Eq. (4b), taken at the equality gives

$$\Psi_s \varphi_{gs} \delta_{max} B \hat{E} / u_1 = \hat{F}_{T_s} \tag{30}$$

which in turn allows δ_{max} to be expressed as

$$\delta_{max} = \frac{u_1 \hat{F}_{T_s}}{\Psi_s \varphi_{gs} B \hat{E}} \tag{31}$$

Serviceability failure occurs if the actual (random) settlement, δ, exceeds the serviceability limit, δ_{max}, so that the failure probability is

$$p_f = \mathbf{P}[\delta > \delta_{max}] = \mathbf{P}\left[F_T\frac{\hat{E}}{E_g} > \frac{\hat{F}_{T_s}}{\Psi_s\varphi_{gs}}\right] \tag{32}$$

which made use of Eq. (31). Note that $F_T, \hat{E}$, and E_g are all random and that Eq. (32) is identical in form to Eq. (11). In fact, if W is now defined as

$$W = F_T\frac{\hat{E}}{E_g} \tag{33}$$

then Eqs. (13)–(15) can be used to determine the failure probability and required consequence and resistance factors for this SLS design problem. The only small differences occur in Eq. (20), which now becomes

$$\sigma^2_{\ln W} = \sigma^2_{\ln F_T} + \sigma^2_{\ln E}\left[\gamma_f + \gamma_s - 2\gamma_{fs}\right] \tag{34}$$

and in the averaging region under the footing shown in Fig. 2, which now becomes $V_f = B \times B \times H$.

Table 3 shows the parameters of the problem used to investigate the consequence and resistance factors required for the SLS design of a shallow foundation. As in the previous ULS example, the choice in mean values is not important to the total resistance factor, $\Psi_s\varphi_{gs}$ – it is the coefficients of variation, v_E and v_{F_T}, and the correlation length θ that are important.

In the study by Fenton et al. [6], the characteristic design load was assumed equal to the median load, $\hat{F}_{T_s} = \exp\left\{\mu_{\ln F_T}\right\}$ in order to simplify the calculation of the total resistance factor in Eq. (15), which becomes

$$\Psi_s\varphi_{gs} = \exp\left\{-\beta\sigma_{\ln W}\right\} \tag{35}$$

However, it may make more sense to choose $\hat{F}_{T_s} = \mu_{F_T}$, in which case Eq. (15) would be used directly. In any case, the difference between the mean and median load is generally negligible, so that the median load can still be used as the characteristic design load while still using Eq. (15).

The remaining issue is the computation of the variance reduction terms in Eq. (21). For this, the averaging region V_f must be known, and it will not be known until after a total resistance factor has been decided upon. The solution is to choose the footing width B equal to some typical value, and the footing width obtained using the median elastic modulus, $E_{med} = \exp\left\{\mu_{\ln E}\right\}$ was found to give good results. The value of E_{med} is shown in Table 3 for the particular example considered. Using this elastic modulus and solving the LRFD Eq. (30) for the design footing width, B, leads to

$$B = u_1\left(\frac{\hat{F}_{T_s}}{\Psi_s\varphi_{gs}\delta_{max}E_{med}}\right) = 0.61\left(1 - e^{-1.18H/B}\right)\left(\frac{\hat{F}_{T_s}}{\Psi_s\varphi_{gs}\delta_{max}E_{med}}\right) \tag{36}$$

Table 3
Parameters used in the investigation of required resistance and consequence factors for the SLS design of shallow foundations.

Parameter	Value
$\mu_E,\ v_E$	30 MPa, 0.3
E_{med}	28.73 MPa
$\mu_{F_T},\ v_{F_T}$	2000 kN, 0.1
$\hat{F}_{T_s}$	1990 kN
$\Delta x,\ H$	0.2 m, 5 m
θ	0.1–50 m
φ_o	0.6

which is non-linear in B and must be solved using an iterative root-finding algorithm. The required footing width can be obtained by 1-pt iteration,

$$B_{i+1} = 0.61\left(1 - e^{-1.18H/B_i}\right)\left(\frac{\hat{F}_{T_s}}{\varphi_o\delta_{max}E_{med}}\right) \tag{37}$$

for $i = 0, 1, \ldots$ with starting guess $B_0 = 0.4\hat{F}_{T_s}/(\varphi_o\delta_{max}E_{med})$. Note that this iteration requires φ_o, which is a reasonable estimate of $\Psi_s\varphi_{gs}$ since the latter is not known until after V_f has been estimated. In this study, a value of $\varphi_o = 0.6$ was found to be reasonable. The iterations stop when subsequent values of B differ by less than some relative error tolerance (0.0001 in this study).

The resistance factors required to achieve a typical lifetime maximum acceptable failure probability of $p_m = 1/500$ for serviceability of a shallow foundation are shown in Fig. 5.

As in Fig. 3, a 'worst case' correlation length is clearly visible in Fig. 5 which, again, is approximately equal to the distance between the footing and the sample location.

The consequence factors shown in Fig. 6 are very similar to those seen for SLS in Fig. 4, suggesting that $\Psi_s \simeq \Psi_u$. This similarity will be seen in all four geotechnical problems considered here.

5. Factors for the ULS design of deep foundations

To study the ULS failure probability, and resulting required resistance factors, of deep foundations, the pile and soil sample configuration shown in Fig. 7 is considered. Now H is the pile length, and D is the depth of the sample. Since this is an ultimate resistance problem, the resistance is due to bearing resistance under the pile tip and due to cohesive and/or frictional resistance along the pile surface. If the tip bearing resistance is ignored, then the ultimate resistance is derived from shear resistance along the pile surface, which means that V_f is a one-dimensional average along the length of the pile. A 1-D average is appropriate if the pile diameter is small relative to the correlation length, which is probably a reasonable assumption. Similarly, the averaging region for the sample will also be assumed here to be one-dimensional, which again is probably reasonable since Δx is generally quite small relative to (at least the worst case) correlation length.

Following the results and methodology presented by Naghibi and Fenton [17], the pile is assumed to be in a cohesive soil (the frictional case is handled similarly) and is designed using the α method (see, e.g., [4]). The design pile length is thus obtained by satisfying the LRFD requirement of Eq. (4a) to give

$$H = \frac{\hat{F}_{T_u}}{\Psi_u\varphi_{gu}p\alpha\hat{c}} \tag{38}$$

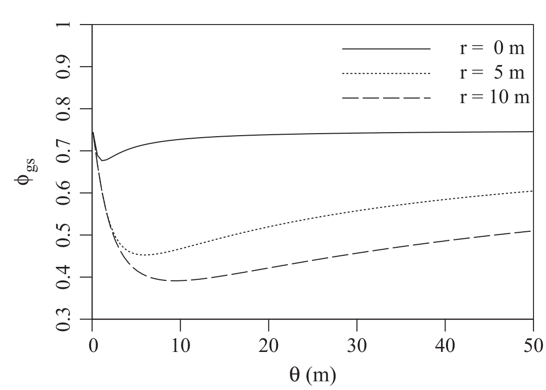

Fig. 5. Resistance factors required to achieve $p_m = 1/500$ ($\beta = 2.9$) for the serviceability design of a shallow foundation ($v_E = 0.3$).

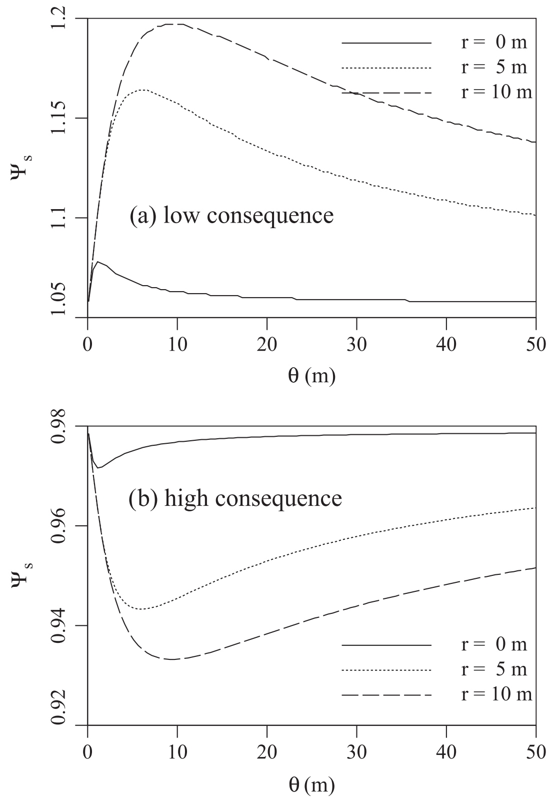

Fig. 6. SLS consequence factors for shallow foundations required to adjust $p_m = 1/500$ ($\beta = 2.9$) to low consequence $p_m = 1/100$ ($\beta = 2.3$) in (a) and to high consequence $p_m = 1/1000$ ($\beta = 3.1$) in (b).

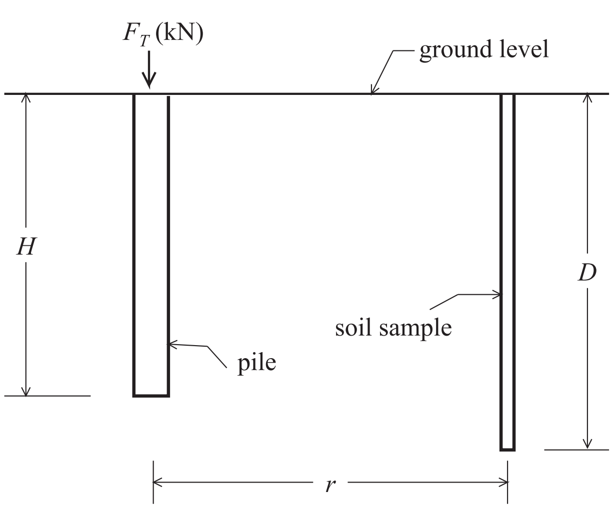

Fig. 7. Relative location of pile and soil sample.

where p is the pile perimeter length, α is the adhesion coefficient, and $\hat{c}$ is the characteristic cohesion along the pile. Although the actual cohesion seen as the average along the pile, c_g, and the sample estimate of that cohesion, $\hat{c}$, are both arithmetic averages, as is appropriate, they are also both deemed to be lognormally distributed. Although, the lognormal distribution is also reasonable, it means that the mean and variance of c_g and $\hat{c}$ have to be approximated. The first order approximations used by Naghibi and Fenton [17] give the same variances as if geometric averages were used, so that Eq. (21) can still be used to provide the needed variance reductions and average correlation for ρ being the correlation coefficient between $\ln c$ at any two points.

It can be seen from Eq. (38) that the pile length, and therefore the averaging region $V_f = H$, depends on the total resistance factor, $(\Psi_u \varphi_{gu})$. This suggests that an iteration might be required, as suggested at the end of the section on "Theoretical failure probability and derived design factors". However, the authors found that this wasn't necessary and that using a fixed length for V_f of

$$V_f = \frac{\hat{F}_{T_u}}{\varphi_o p \alpha \mu_c} \tag{39}$$

gave reasonably accurate failure probability predictions, in comparison to RFEM simulations. In Eq. (39), φ_o is a moderate resistance factor whose value is taken to be 0.7. The remaining parameters considered for this deep foundation ULS design problem are shown in Table 4. Note that in the failure probability for this case,

$$p_f = \mathbf{P}\left[F_T \frac{\hat{c}}{c_g} > \frac{\hat{F}_{T_u}}{\Psi_u \varphi_{gu}}\right] = \mathbf{P}\left[\ln W > \ln\left(\hat{F}_{T_u} \Psi_u \varphi_{gu}\right)\right]$$
$$= 1 - \Phi\left(\frac{\ln\left(\hat{F}_{T_u}/\Psi_u \varphi_{gu}\right) - \mu_{\ln F_T}}{\sigma_{\ln F_T}}\right) \tag{40}$$

the parameters p and α cancel out and so are not needed to estimate $\Psi_u \varphi_{gu}$.

Fig. 8 shows the resistance factors required to achieve $p_m = 1/5000$ for various sampling distances (site understanding) and correlation lengths. As usual, there is a distinct 'worst case' correlation length which is approximately equal to the distance to the sample location, r. When r is equal to zero, the pile and sample are at the same location. This means that the sample will basically be an excellent to identical indicator of the cohesion along the pile so that the failure probability will go to zero or the resistance factor will go to 1.0. The only reason that the resistance factor shows a little dip below 1.0 is because the sample length,

Table 4
Parameters used in the investigation of required resistance and consequence factors for the ULS design of deep foundations.

Parameter	Value
$\mu_c,\ v_c$	50 kPa, 0.3
$\mu_L,\ v_L$	20 kN, 0.3
$\mu_D,\ v_D$	60 kN, 0.15
$\hat{F}_{T_u}$	131 kN
$\mu_{F_T},\ v_{F_T}$	80 kN, 0.14
D	4 m
θ	0.1–50 m
φ_o	0.7

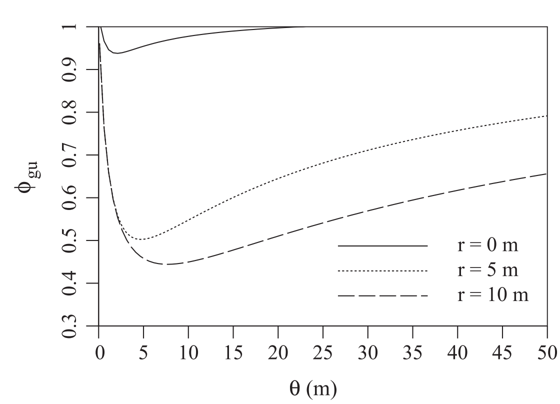

Fig. 8. Resistance factors required to achieve $p_m = 1/5000$ ($\beta = 3.5$) for the ULS design of a deep foundation.

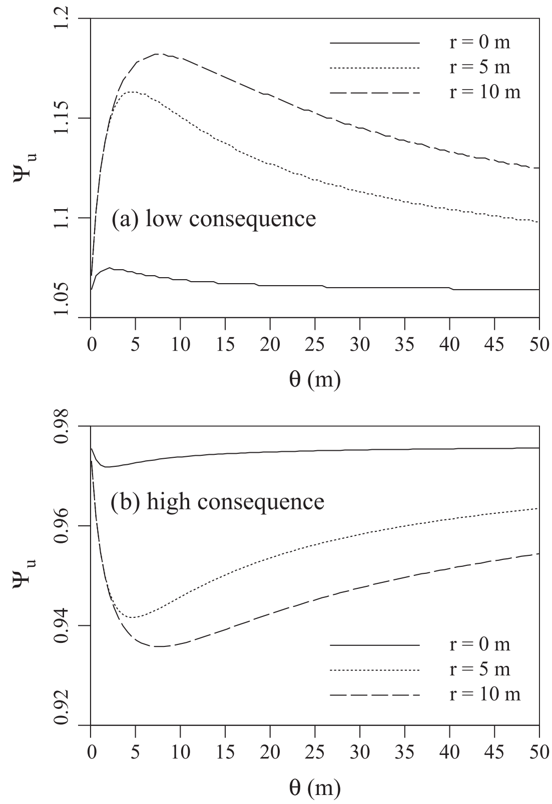

Fig. 9. ULS consequence factors for deep foundations required to adjust $p_m = 1/5000$ ($\beta = 3.5$) to low consequence $p_m = 1/1000$ ($\beta = 3.1$) in (a) and to high consequence $p_m = 1/10,000$ ($\beta = 3.7$) in (b).

$D = 4$ m, is not generally the same as the pile length, H, which means that the sample will not generally be precisely equal to the actual cohesion along the pile. If the $r = 5$ m case is considered, the worst case correlation length suggests that $\varphi_{gu} = 0.5$ should be used in design.

Fig. 9 illustrates the consequence factors required to shift the target maximum acceptable failure probability, p_m, for low and high failure consequences. Again, the factors are virtually identical to those seen in Figs. 6 and 4 for SLS and ULS design of shallow foundations and $\Psi_u = 0.9$ is conservative for high failure consequences while $\Psi_u = 1.15$ is reasonable for low failure consequences.

6. Factors for the SLS design of deep foundations

The last geotechnical problem considered in this paper is the SLS design of deep foundations. The results presented here are taken from the work presented by Naghibi et al. [15,16] and the basic pile/sample geometry is identical to that shown in Fig. 7. Since this is a serviceability design problem, the ground is represented as a spatially variable elastic modulus field and the required size of the averaging regions was estimated using the random finite element method (RFEM), just as was done for the SLS design of shallow foundations previously. Despite the similarity between the problems, the SLS design of deep foundations was much more difficult than any of the other problems for two reasons:

(1) the averaging volume V_f was not as clearly defined as it was for shallow foundations. For shallow foundations, it is pretty clear that settlement directly compresses the region below the foundation and choosing $V_f = B \times B \times H$ (where H is depth to bedrock) seems natural. For deep foundations, however, the settlement of the pile is due to both elastic deformation of the pile itself along with the deformation of the soil surrounding the pile. There was no a priori natural averaging domain for V_f and it had to be found by trial-and-error.
(2) in the SLS design of a shallow foundation, the settlement can be reduced to zero simply by increasing the foundation area indefinitely (flexibility of the footing itself was ignored). However, for a given pile, there is a limit to how small the settlement can be made as the pile length is increased. This is because when the pile becomes too long, its own deformation becomes the main contributor to the settlement at the pile head. In other words, when the load is too large, or the maximum settlement, δ_{max}, is too small there may not be a pile length H which satisfies the requirement that $\delta < \delta_{max}$ – multiple piles may be required. If this happens, the overall reliability analysis becomes very much more complicated. See Naghibi et al. [15] for an analytical solution to this problem.

In this paper only the design of a single pile is considered, which means that attention is restricted to those load, maximum displacement, and pile attributes (diameter and elastic modulus) combinations that only require a single pile to avoid entering the serviceability limit state.

Table 5 shows the parameters of the problem used to investigate the consequence and resistance factors required for the SLS design of a deep foundation. As usual, the choice in mean values is not important to the total resistance factor, $\Psi_s\varphi_{gs}$, which depends primarily on the coefficients of variation, v_E and v_{F_T}, and on the correlation length θ.

The pile settlement is given by Naghibi et al. [16] to be

$$\delta = \frac{F_T}{E_g d} I_p \tag{41}$$

where I_p is the settlement influence factor. Naghibi et al. [16] developed a regression for I_p,

$$I_p = \left[a_0 + \frac{1}{(H/d + a_1)^{a_2}} \right] \tag{42}$$

where the regression coefficients a_0, a_1 and a_2 are functions of the pile to soil stiffness ratio. Replacing E_g with the characteristic $\hat{E}$, and setting $\delta = \delta_{max}$ leads to the following LRFD requirement from Eq. (4b),

$$\Psi_s \varphi_{gs} \left(\frac{\delta_{max} d\hat{E}}{I_p} \right) \geqslant \hat{F}_{T_s} \tag{43}$$

which, using Eq. (42), can be inverted to solve for the pile length, H, required to have the predicted settlement (using the characteristic elastic modulus) just equal δ_{max},

$$H = d \left[\left(\frac{1}{\left(\Psi_s \varphi_{gs} \delta_{max} d\hat{E}/\hat{F}_{T_s} \right) - a_0} \right)^{1/a_2} - a_1 \right] \tag{44}$$

Table 5
Parameters used in the investigation of required resistance and consequence factors for the SLS design of deep foundations.

Parameter	Value
μ_E, v_E	30 MPa, 0.3
μ_{F_T}, v_{F_T}	1600 kN, 0.1
$\hat{F}_{T_s}$	1600 kN
δ_{max}	0.025 m
Δx, D	0.3 m, 10 m
θ	0.1–50 m

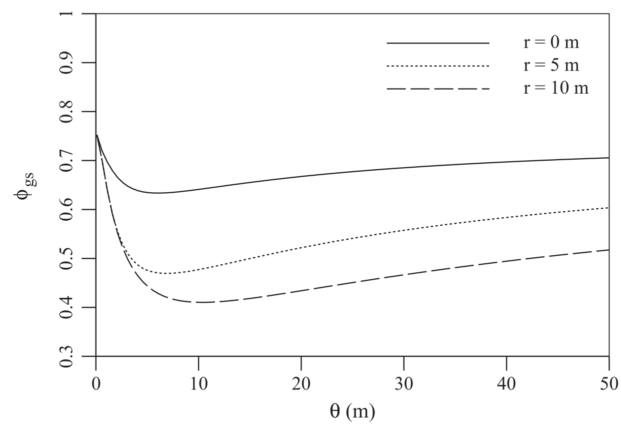

Fig. 10. Resistance factors required to achieve $p_m = 1/500$ ($\beta = 2.9$) for the serviceability design of a deep foundation.

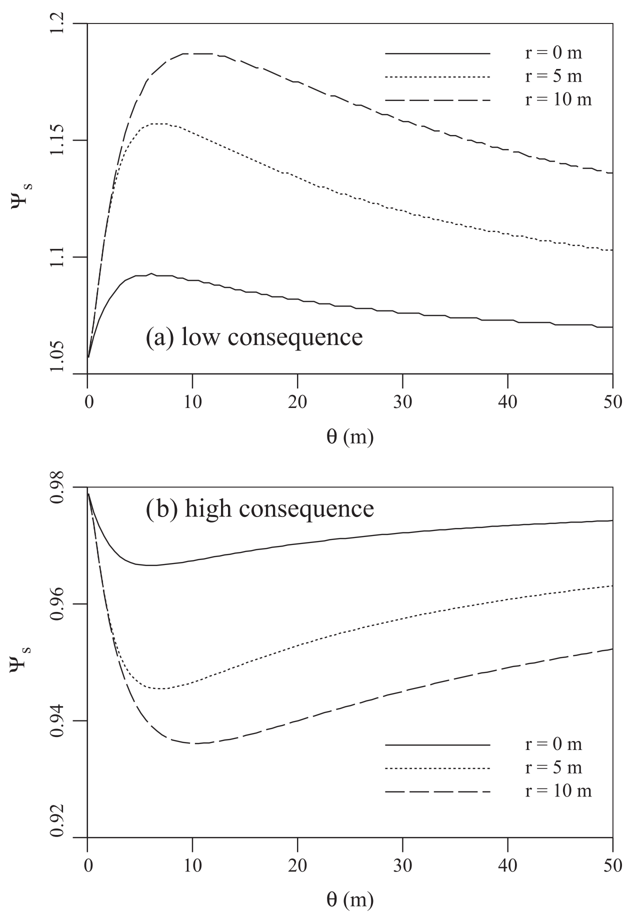

Fig. 11. SLS consequence factors for deep foundations required to adjust $p_m = 1/500$ ($\beta = 2.9$) to low consequence $p_m = 1/100$ ($\beta = 2.3$) in (a) and to high consequence $p_m = 1/1000$ ($\beta = 3.1$) in (b).

The probability of failure, p_f, the definition of W, and the variance of $\ln W$ are as given for the SLS design of shallow foundations in Eqs. (32)–(34), which in turn are just special cases of the unified theory given in the section entitled "Theoretical failure probability and derived design factors". The main complication, as mentioned above, was determining the appropriate averaging volume, V_f. After much trial-and-error an averaging volume of $V_f = 2 \times 2 \times 2H$ was found to give the best results. What this means is that a rectangular 'tube' of soil having width 2 m and depth equal to twice the pile length (also in metres) was found to be the volume of 'actively' deforming soil around the pile.

Having established $V_f = 2 \times 2 \times 2H$, an iteration was required to determine $(\Psi_s \varphi_{gs})$ since the pile length H is dependent on the resistance factor (see Eq. (44)). The determination of $(\Psi_s \varphi_{gs})$ via Eq. (15) was carried out using a bisection algorithm to find the root, with respect to $(\Psi_s \varphi_{gs})$, of $p_f - p_m = 0$. In other words, using Eq. (14), the value of $(\Psi_s \varphi_{gs})$ which satisfies

$$\Phi\left(\frac{\ln\left(\Psi_s \varphi_{gs}\right) + \mu_{\ln W} - \ln \hat{F}_T}{\sigma_{\ln W}}\right) = p_m \tag{45}$$

was found by bisection, bearing in mind that H is required to find $\sigma_{\ln W}$ via Eqs. (44), (21) and (20). The results, for the typical consequence level ($p_m = 1/500$) are shown in Fig. 10.

Besides the presence again of a worst case correlation length in Fig. 10, it is of note that even when $r = 0$, the resistance factor is quite low, around $\varphi_{gs} = 0.7$. This is because, although the sample is taken at the pile location, there is a very large difference between V_f and V_s. Even if the sample happened to be of the same length as the pile, the sample has cross-sectional area of 0.09 m^2, whereas the averaging volume around the pile includes much more variation, having a cross-sectional area of 4 m^2.

Fig. 11 shows the consequence factors required in the case of the SLS design of deep foundations. These are virtually identical to those seen earlier and so do not need further discussion. In addition, the use of a consequence factor of $\Psi_s = 0.9$ remains conservative for the high failure consequence case, and $\Psi_s = 1.15$ remains reasonable for the low failure consequence case.

7. Conclusions

The paper presents a unified theory which allows the estimation of both failure probability and the resistance and consequence factors required to achieve a target failure probability. Perhaps the most important component of this unified theory is Eq. (20), which in a more generalized form appears as

$$\sigma^2_{\ln W} = \sigma^2_{\ln F_T} + \sigma^2_{\ln R}\left[\gamma_f + \gamma_s - 2\gamma_{fs}\right] \tag{46}$$

where R denotes 'resistance' and is replaced by the ground parameter(s) which are important for the problem. This equation can then be used in Eq. (14), to determine failure probability, or in Eq. (15) to determine required resistance factors given a target reliability. Eq. (46) includes the following components;

(1) variability of the applied load ($\sigma_{\ln F_T}$),
(2) variability of the ground ($\sigma_{\ln R}$),
(3) variance reduction due to averaging of the ground properties under and around the foundation (γ_f),
(4) variance reduction due to averaging of the ground properties found in the soil sample (γ_s), and perhaps most importantly,
(5) correlation between the sample and the properties of the ground under and around the foundation (γ_{fs}).

The last allows for a reasonable modeling of 'site understanding' so that resistance factors can be selected based on how well the response of the ground supporting the foundation can be predicted. The distance r used in this study can be used as a proxy to reflect general site and model understanding, where 'model understanding' refers to how accurate the ground response prediction model is. As site and model understanding decreases, the corresponding value of r selected in this study would be increased.

The 'typical' resistance factors, φ_{gu} and φ_{gs}, are similar to those in current practice for ULS design, but are significantly lower than those in current practice for SLS design. It is believed that the reason that SLS design uses resistance factors of 1.0 (or close to 1.0) is because traditional approaches to estimating foundation settlement have their own built in conservatism. The use of the SLS resistance factors suggested in this paper presume the use of a more accurate *unbiased* estimate of the mean settlement – the resistance factor then takes care of the variability around that mean.

The consequence factor is used to adjust the target failure probability from the 'typical' level to either a high or low consequence level. So, for example, the high consequence factor for ULS is adjusting $p_m = 1/5000$ to $p_m = 1/10,000$, i.e., dividing the target failure probability in half. The high consequence factor for SLS is adjusting $p_m = 1/500$ to $p_m = 1/1000$, again dividing the target failure probability in half. Because the relative change in target probability is the same, it is reasonable to expect that the consequence factors will be similar between SLS and ULS. A review of Figs. 4, 6, 9 and 11 shows that the consequence factors are very similar over all four geotechnical problems, meaning that they are largely independent of the limit state under consideration, so long as the target maximum acceptable failure probabilities are scaled similarly. Thus, the distinction between Ψ_s and Ψ_u can be dropped, and a common consequence factor, Ψ, used. The smaller the consequence factor, the lower the target failure probability, p_m. Thus, for high failure consequence systems, a Ψ value of 0.9 is recommended, since this was seen to be conservative ($p_f < p_m$) for all cases considered.

For low failure consequence systems, it is not necessary to be quite so conservative (because of the low failure consequences), so a value of $\Psi = 1.15$ appears to be reasonable. This is slightly unconservative for high site understanding ($r = 0$ m) but about right for typical site understanding ($r = 5$ m) and somewhat conservative for low site understanding ($r = 10$ m).

Finally, it is noted that the use of a separate variable resistance factor, to account for uncertainty in the ground response, and a variable consequence factor, to adjust the target reliability, is a concept that has now been adopted in the 2014 edition of the Canadian Highway Bridge Design code [3].

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