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ARTICLE

Probabilistic considerations for the design of deep foundations against excessive differential settlement

Farzaneh Naghibi, Gordon A. Fenton, and D.V. Griffiths

Abstract: Current foundation design practice for serviceability limit states involves proportioning the foundation to achieve an acceptably small probability that the foundation settlement exceeds some target maximum total settlement. However, it is usually differential settlement that leads to problems in the supported structure. The design question, then, is how should the target maximum total settlement of an individual foundation be selected so that differential settlement is not excessive? Evidently, if the target maximum total settlement is increased, the differential settlement between foundations will also tend to increase, so that there is a relationship between the two, although not necessarily a simple one. This paper investigates how the target maximum total settlement specified in the design of an individual foundation relates to the distribution of the differential settlement between two identical foundation elements, as a function of the ground statistics and the distance between the two foundations. A probabilistic theory is developed, and validated by simulation, which is used to prescribe target maximum settlements employed in the design process to avoid excessive differential settlements to some acceptable probability.

Key words: foundation settlement, deep foundations, differential settlement, finite element method, design pile length.

Résumé : La pratique courante de conception de fondations avec des états limites de service consiste à proportionner la fondation à parvenir à une suffisamment faible probabilité que le tassement des fondations dépasse le tassement total maximum cible. Cependant, il est généralement le tassement différentiel qui provoque des problèmes dans la structure supportée. La question sur la conception, alors, est de savoir comment le tassement total maximum cible d'une fondation individuelle doit être choisi de telle sorte que le tassement différentiel ne soit pas excessif. Évidemment, si le tassement total maximum cible est augmenté, le tassement différentiel entre les fondations aura aussi tendance à augmenter, de sorte qu'il existe une relation entre les deux, mais pas nécessairement une relation simple. Cet article examine la façon dont le tassement total maximum cible spécifié dans la conception d'une fondation se rapporte à la distribution du tassement différentiel entre deux éléments de fondation de base identiques, en fonction des statistiques sur le sol et la distance entre les deux fondations. Une théorie probabiliste est développée et validée par simulation, qui est utilisée pour prescrire des tassements maximaux cibles utilisés dans le processus de conception pour éviter les tassements différentiels excessifs à une certaine probabilité acceptable. [Traduit par la Rédaction]

Mots-clés : tassement de fondations, fondations profondes, tassement différentiel, méthode des éléments finis, conception de longueur de pieux.

Introduction

Geotechnical foundation design is often governed by serviceability limit states (SLSs), relating to settlement, rather than by ultimate limit states (ULSs), which relate to safety. Most modern geotechnical design codes state that the serviceability limit state can be avoided by designing each foundation to settle by no more than a specified maximum tolerable settlement, δ_{max}. However, in the case of foundations, it is usually differential settlements that govern the serviceability of the supported structure. For example, if all of the foundations of a supported structure settle equally, but excessively, then the approaches to the structure may have to be modified, but the structure itself will not suffer from either a loss of serviceability or a loss of safety.

Almost certainly, individual foundations will not settle equally so that the differential settlement between foundations can lead to loss of serviceability and even catastrophic ultimate limit state failure in the supported structure. So the question is, how should differential settlement between foundations be properly accounted for in the foundation design process?

Although the settlement of deep foundations is not generally a concern if the piles are driven to refusal, settlement can become a design issue if no stiff substratum is encountered. As a result, this paper will concentrate attention on piles that are not end-bearing; that is, on piles whose settlement resistance is derived from skin friction and (or) adhesion with the surrounding soil.

Design code provisions should be kept as simple as possible, while still achieving a target reliability with respect to both serviceability and ultimate limit states. This means that design codes should retain their maximum total settlement requirements, but the specified maximum settlement should be reviewed to reasonably confirm that differential settlements do not result in achieving either serviceability or ultimate limit states in the supported structure.

This paper investigates how the maximum settlement specified in a design code for an individual foundation relates to the distri-

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bution of differential settlement between two foundations, as a function of the ground statistics and the distance between the two foundations. Figure 1 illustrates the settlement of two piles founded in a spatially variable ground. The paper will propose design code requirements on maximum settlements for individual foundations that aim to achieve target reliabilities against excessive differential settlements between pairs of foundations.

In this paper, the settlement of a pair of floating piles founded in a three-dimensional (3-D) spatially random soil mass, each supporting a vertical load, F_T, is studied using the random finite element method (RFEM, Fenton and Griffiths 2008) employing a linearly elastic model. It is emphasized that this paper is about differential settlement, not total settlement. The mean total settlement entirely disappears from the predicted differential settlement, which depends only on the variance — the mean differential settlement is zero. This implies that the linearly elastic prediction used here for the mean settlement of each pile can be replaced by a more sophisticated nonlinear model without changing the results of this paper, as long as the more sophisticated model properly accounts for spatial variability in the ground. In turn, the latter means that it is essential to use a settlement model that properly accounts for spatial variability. While total settlement may be better predicted using a nonlinear model, with myriad parameters, the resulting model is of no use in this study unless the spatial statistics (mean, variance, and spatial correlation structure) can be estimated for the various parameters used in the more sophisticated model. However, it is difficult enough to obtain the spatial statistics of a simple scalar random field, such as the elastic modulus field, without complicating things by trying to make use of spatially varying multi-variate, possibly cross-correlated, random processes that would be associated with a nonlinear settlement model. In other words, the value of this paper would be muddied by trying to employ a model of unrealistic probabilistic complexity. The elastic model is perfectly adequate to capture the effects of spatial variability, especially since the mean settlement cancels out, so that errors in its estimation disappear.

With these thoughts in mind, a probabilistic model for differential settlement is presented, which is then validated via Monte Carlo simulation. The results are used to propose design provisions for piles to avoid excessive differential settlement at a target reliability level.

The paper is organized as follows. In the "Finite element model" section, a finite element (FE) model is presented for a pair of floating piles founded in a 3-D spatially random soil mass, each supporting a vertical load. A theoretical approach to estimating the distribution of differential pile settlement is developed in the "Probabilistic settlement model" section, and the approach is validated via simulation in the "Validation of theory via Monte Carlo simulation" section. Design code recommendations are then presented in the "Design recommendations" section, followed by conclusions and proposed future work in the final section.

Finite element model

The random settlement of a single pile, which was studied in depth by Naghibi et al. (2014*b*), is highly dependent on the random elastic modulus field of the surrounding soil, as well as on the pile geometry. In addition, when settlement of pile groups is of interest, mechanical interaction between the piles plays an important role in both total settlement and differential settlement.

Consider two neighboring piles of identical geometry, supporting loads F_{T_1} and F_{T_2} and separated by distance s, as depicted in Fig. 2. For generality in developing the theory, the pile loads will be considered to be random and possibly correlated (although in the validation and design sections, as well as the calibration of the mechanical interaction factor, it is assumed that $F_{T_1} = F_{T_2} = F_T = \mu_T$, i.e., nonrandom). If δ'_1 is the settlement of a vertically

Fig. 1. Slice through a random finite element method (RFEM) mesh of a ground supporting two piles.

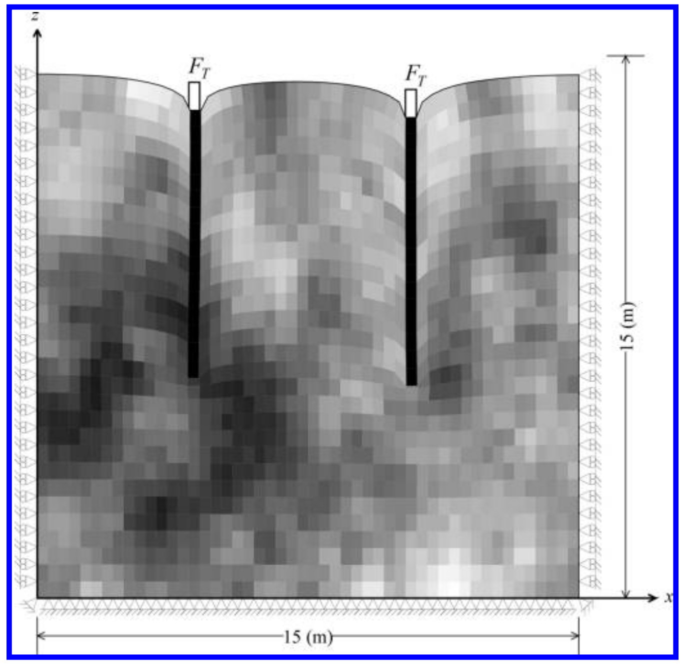
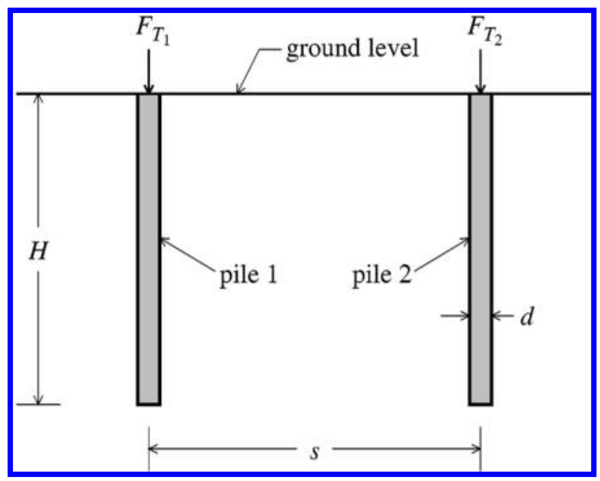


Fig. 2. Relative location of two piles.



loaded individual pile without any neighboring piles, and δ_1 is the overall settlement of the pile due to its loading and due to settlement of a neighboring pile, δ'_2, then

$$\delta_1 = \delta'_1 + \eta\delta'_2 \quad (1)$$

where η is the mechanical interaction factor between the two piles, which is a function of pile spacing and pile length. Rearranging eq. (1) and solving for η gives

$$\eta = \frac{\delta_1 - \delta'_1}{\delta'_2} \quad (2)$$

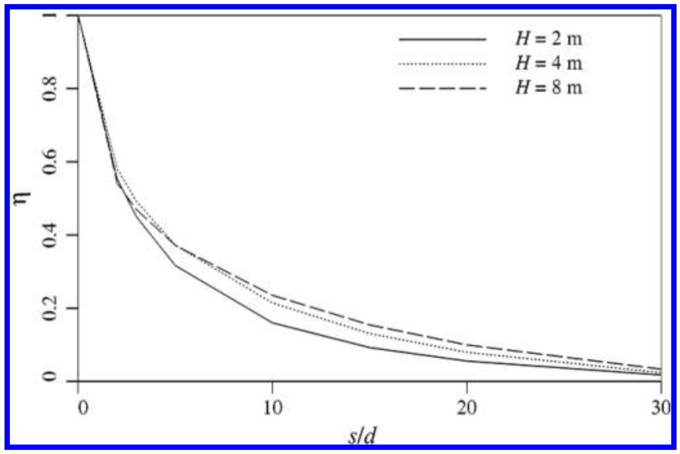
To predict η, predictions (or observations) of δ_1, δ'_1, and δ'_2 are needed. In this paper, these quantities will be found using a linear elastic FE model of the soil (Smith et al. 2014) with deterministic elastic modulus field ($E = \mu_E$ everywhere, where E is the elastic modulus and μ_E is its mean). The piles are founded in a 3-D linear elastic soil mass modeled using a 50 × 30 × 30 FE mesh, as illustrated in Fig. 1. Eight-node brick elements are used with dimensions 0.3 m × 0.3 m in the *x*, *y* (plan) and by 0.5 m in the *z* (vertical) directions. Within the mesh, piles are modeled as columns of elements having depth *H*, and hence have a square cross section with dimension *d* = 0.3 m.

The prediction of η is done for three pile lengths, *H* = 2, 4, and 8 m, with separation distance, *s*/*d*, ranging between 2 and 30, where *s* is the center-to-center pile spacing and *d* is the pile diameter. For a particular choice of pile length, *H*; pile spacing, *s*; applied load, F_T = 2.16 MN; pile to soil stiffness ratio, $k = E_p/E = 700$; and soil elastic modulus, *E* = 30 MPa; three FE analyses are performed: one with pile 1 only, one with pile 2 only, and one with both piles 1 and 2 separated by distance *s*. The piles are placed at elements (50 – *s*/*d*)/2 and (50 + *s*/*d*)/2 numbered in the *x*-direction. For example, the pile 1 only case involves the FE analysis of single pile settlement where pile 1 is placed at element (50 – *s*/*d*)/2, while the two pile case involves the FE analysis of two neighboring piles, piles 1 and 2, placed at elements (50 – *s*/*d*)/2 and (50 + *s*/*d*)/2, respectively.

Note that the mechanical interaction, η, depends on pile spacing, s, and pile length, *H*, and is independent of *E* or F_T. The dependence on Poisson's ratio, ν, is negligible. The piles are placed at least 10 elements away from the boundaries of the FE model, which leads to a relative pile settlement error of less than 10% (Naghibi et al. 2014*a*), so that the influence of boundary conditions on pile settlement is deemed to be negligible.

The resulting sequence of FE analyses provided predicted values of η for various *s*/*d* and pile lengths, *H*, as shown in Fig. 3. It is evident that η increases with increasing pile length, *H*, and decreases with increasing *s*/*d*, as expected. For values of *H* other than those specified in Fig. 3, η is predicted using linear interpolation for $2 \leq H \leq 8$ m and linear extrapolation when *H* falls outside this range.

Fig. 3. Plot of interaction factor, η, using FE model for F_T = 2.16 MN, *k* = 700, ν = 0.3, and *E* = 30 MPa.



Probabilistic settlement model

Attention is now turned to a probabilistic model of pile settlement, where the soil is assumed to be a spatially variable random field. To estimate the pile settlement, it is first assumed that the soil surrounding the pile is perfectly bonded to the pile shaft through friction and (or) adhesion. Any displacement of the pile is thus associated with an equivalent displacement of the adjacent soil. Following the classic work of authors such as Poulos and Davis (1980), Randolph and Wroth (1978), and Vesic (1977), the soil is assumed to be linearly elastic, so that this displacement is resisted by a force that is proportional to the soil's elastic modulus and the magnitude of the displacement. Thus, the support provided by the soil to the pile depends on the elastic properties of the surrounding soil.

To design a pile against entering the serviceability limit state — that is, against entering a failure state where the pile's actual settlement exceeds a maximum tolerable settlement — a settlement prediction model is required. If the model is good, then it will provide a good estimate of the mean pile settlement and the in situ actual pile settlement will be due to natural "residual" soil variability around the predicted mean. The settlement prediction model is used to determine the pile design such that the predicted mean settlement is some fixed fraction (specified by the load and resistance factors) of the maximum tolerable settlement. If the settlement prediction model is poor, then it also contributes to the variability in the prediction of the actual settlement. This source of variability will be referred to here collectively as the "degree of site and prediction model understanding", which includes (*i*) the degree of understanding of the ground properties and geotechnical properties throughout the site and (*ii*) the accuracy and degree of confidence about the numerical performance prediction model used to estimate the serviceability geotechnical resistances.

It is assumed in this paper that a sufficiently accurate settlement prediction model is used for the pile design, so that model error itself is attributable only to errors in the soil parameters used in the model; that is, to the degree of site understanding. This is probably a reasonable assumption, because if the (possibly nonlinear) properties of the soil through which the pile passes, along with the nature of the interface between the pile and the soil, are all well known, then models do exist that can provide very good estimates of the mean pile settlement. This paper is not attempting to provide an improved settlement prediction model. In fact, a decision about the degree of site and prediction model understanding used in the pile design process is left to the designer. This paper concentrates on the residual settlement variability (around the mean) after the design has been performed. It is assumed that this variability arises from the spatial variability of the soil itself, along with uncertainty in the soil property estimates used in the prediction model.

It is recognized that pile settlement is almost certainly nonlinear, and so the elastic modulus mean used in this simulation must be considered to be a secant modulus, which approximates the curved nature of the actual pile load–settlement curve. However, the details of the mean settlement predictor used to design a pile and (or) to estimate the distribution of differential pile settlement are not important to the subsequent probabilistic analysis (which is relative to the mean) and, of course, the reader is encouraged to use the best settlement prediction available to them. The linear model used in this paper is, however, the best currently available to predict the effects of spatial variability of the soil on the distributions of settlement and differential settlement.

The spatially varying elastic modulus field, which is assumed here to have a constant Poisson's ratio, ν, may be characterized by an equivalent soil elastic modulus, E_g. The equivalent elastic modulus is a spatially uniform value that yields the same settlement as the pile experiences in the actual spatially varying soil (Fenton and Griffiths 2007). E_g will be assumed here to be the geometric average of the spatially varying elastic modulus field, *E*, as will be discussed shortly. The elastic modulus is assumed to be lognormally distributed with mean μ_E, standard deviation σ_E, and spatial correlation length, $\theta_{\ln E}$. The lognormal distribution is commonly used to represent non-negative soil properties and means that ln *E* is normally distributed with parameters $\mu_{\ln E}$ and $\sigma_{\ln E}$. The distribution parameters of ln *E* can be obtained from the mean and standard deviation of *E* using the following transformations:

$$
\begin{aligned}
\mu_{\ln E} &= \ln(\mu_E) - \frac{1}{2}\sigma^2_{\ln E} \\
\sigma^2_{\ln E} &= \ln(1 + v_E^2)
\end{aligned} \tag{3}
$$

where $v_E = \sigma_E/\mu_E$ is the coefficient of variation of E. If the soil's elastic modulus, E, is lognormally distributed, as assumed, then E_g will also be lognormally distributed because geometric averages preserve the lognormal distribution (Fenton and Griffiths 2008).

The correlation coefficient between the log elastic modulus at two points is defined by a correlation function, $\rho_{\ln E}(\tau)$, in which τ is the distance between the two points. In this study, a simple isotropic exponentially decaying (Markovian) correlation function will be employed, having the form

$$
\rho_{\ln E}(\tau) = \exp\left(-\frac{2|\tau|}{\theta_{\ln E}}\right) \tag{4}
$$

where τ is the distance between any two points in the field and $\theta_{\ln E}$ is the correlation length (Fenton and Griffiths 2008).

As the soil is a spatially variable random field, the pile settlement will also be random. Assuming that the pile settlement is approximately lognormally distributed (as was shown to be reasonable by Naghibi et al. 2014*b*), then the task is to find the parameters of that distribution and the distribution of the resulting differential settlement. Assuming further that δ_1 and δ_2 are the total settlements of the two piles shown in Figs. 1 and 2, then δ_1 and δ_2 are identically and lognormally distributed random variables.

The differential settlement between two piles is defined here to be $\Delta = \delta_1 - \delta_2$. If the elastic modulus field is statistically stationary, as assumed here, then the mean differential settlement, μ_Δ, is zero. The mean absolute differential settlement can be approximated by (if Δ is approximately normally distributed)

$$
\mu_{|\Delta|} \approx \sqrt{\frac{2}{\pi}}\sigma_\Delta \tag{5}
$$

which indicates that the mean of the absolute differential settlement is directly related to the standard deviation of Δ, and hence related to the variability of the elastic moduli surrounding the piles. The approximation in eq. (5) is exact if Δ is normally distributed (Papoulis 1991) and, as will be shown shortly, this approximation is in reasonable agreement with simulation-based results.

Investigations by Fenton and Griffiths (2002) suggest that the equivalent elastic modulus as seen by a shallow foundation is a geometric average of the soil's elastic modulus under the foundation. Naghibi et al. (2014*b*) similarly assume that the equivalent elastic modulus, E_g, as seen by a pile is a geometric average of the soil's elastic modulus over some volume, V_f, surrounding the pile

$$
\begin{aligned}
E_g &= \exp\left[\frac{1}{V_f}\int_{V_f} \ln E(\underset{\sim}{x})\, d\underset{\sim}{x}\right] \\
&= \exp\left[\frac{1}{B^2 C}\int_0^B\int_0^B\int_0^C \ln E(x,y,z)\, dz\, dy\, dx\right]
\end{aligned} \tag{6}
$$

where $E(\underset{\sim}{x}) = E(x, y, z)$ is the elastic modulus of the soil at spatial position (x, y, z). The pile is centered on the volume $V_f = B \times B \times C$, where C is measured in the vertical (z) direction.

The settlement of a single pile can then be expressed as

$$
\delta_i' = \delta_{\det}\left(\frac{\mu_E}{E_g}\right)\left(\frac{F_{T_i}}{\mu_T}\right) \tag{7}
$$

where the subscript i is either 1 or 2, and $\delta_{\det}$ is the deterministic settlement of a single pile obtained from a single FE analysis of the problem using $F_{T_i} = \mu_T$ and $E = \mu_E$ everywhere. Substituting eq. (7) into eq. (1) leads to pile settlements, δ_1 and δ_2, as follows:

$$
\begin{aligned}
\delta_1 &= \delta_{\det}\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_1}}{E_{g_1}} + \frac{\eta F_{T_2}}{E_{g_2}}\right) \\
\delta_2 &= \delta_{\det}\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_2}}{E_{g_2}} + \frac{\eta F_{T_1}}{E_{g_1}}\right)
\end{aligned} \tag{8}
$$

The differential settlement, $\Delta = \delta_1 - \delta_2$, between two piles becomes

$$
\Delta = \delta_{\det}(1-\eta)\left(\frac{\mu_E}{\mu_T}\right)\left(\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right) \tag{9}
$$

The variance of Δ is therefore

$$
\sigma^2_\Delta = \delta^2_{\det}(1-\eta)^2\left(\frac{\mu_E}{\mu_T}\right)^2 \mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] \tag{10}
$$

where

$$
\mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] = \mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}}\right] + \mathrm{Var}\left[\frac{F_{T_2}}{E_{g_2}}\right] - 2\mathrm{Cov}\left[\frac{F_{T_1}}{E_{g_1}}, \frac{F_{T_2}}{E_{g_2}}\right] \tag{11}
$$

Now if $X_i = F_{T_i}/E_{g_i}$; $i = 1, 2$; and F_{T_i} and E_{g_i} are lognormally distributed; then X_i is also lognormally distributed. In this case, $\ln X_i = \ln F_{T_i} - \ln E_{g_i}$ is normally distributed with parameters (Naghibi et al. 2014*b*)

$$
\begin{aligned}
\mu_{\ln X_i} &= \mu_{\ln F_T} - \mu_{\ln E_g} = \mu_{\ln F_T} - \mu_{\ln E} \\
&= \ln(\mu_T) - \frac{1}{2}\sigma^2_{\ln F_T} - \ln(\mu_E) + \frac{1}{2}\sigma^2_{\ln E} \\
\sigma^2_{\ln X_i} &= \sigma^2_{\ln F_T} + \sigma^2_{\ln E_g} = \sigma^2_{\ln F_T} + \sigma^2_{\ln E}\gamma_f
\end{aligned} \tag{12}
$$

assuming that F_{T_i} and E_{g_i} are independent, and where γ_f is the variance reduction due to averaging $\ln E$ over the 3-D volume $V_f = B \times B \times C$ surrounding the piles. In detail,

$$
\gamma_f = \frac{1}{V_f^2}\int_0^{V_f}\int_0^{V_f} \rho_{\ln E}(\underset{\sim}{x}_1 - \underset{\sim}{x}_2)\, d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2 \tag{13}
$$

where $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ are two spatial positions within V_f. Note that γ_f is essentially just the average correlation coefficient between all points within the volume V_f.

The distribution parameters of X_i can be obtained from the mean and standard deviation of $\ln X_i$ using the following transformations:

$$
\begin{aligned}
\mu_{X_i} &= \exp\left(\mu_{\ln X_i} + \frac{1}{2}\sigma^2_{\ln X_i}\right) \\
\sigma^2_{X_i} &= \mu^2_{X_i}(e^{\sigma^2_{\ln X_i}} - 1)
\end{aligned} \tag{14}
$$

Using eq. (12) in eq. (14) results in

$$
\begin{aligned}
\mu_{X_i} &= \mathrm{E}\left[\frac{F_{T_i}}{E_{g_i}}\right] = \exp\left[\ln(\mu_T) - \frac{1}{2}\sigma^2_{\ln F_T} - \ln(\mu_E) + \frac{1}{2}\sigma^2_{\ln E} + \frac{1}{2}\sigma^2_{\ln E}\gamma_f\right] \\
&= \frac{\mu_T}{\mu_E}\sqrt{(1+v_E^2)^{\gamma_f+1}} \\
\sigma^2_{X_i} &= \mathrm{Var}\left[\frac{F_{T_i}}{E_{g_i}}\right] = \left[\mu_T^2(1+v_E^2)^{\gamma_f+1}/\mu_E^2\right]\left[\exp\left(\sigma^2_{\ln F_T} + \sigma^2_{\ln E}\gamma_f\right) - 1\right] \\
&= \frac{\mu_T^2}{\mu_E^2}(1+v_E^2)^{\gamma_f+1}\left[\left(1+v_T^2\right)\left(1+v_E^2\right)^{\gamma_f} - 1\right]
\end{aligned} \tag{15}
$$

where $v_E = \sigma_E/\mu_E$ is the coefficient of variation of the elastic modulus field, E.

The covariance between the two lognormal random variables $X_1 = F_{T_1}/E_{g_1}$ and $X_2 = F_{T_2}/E_{g_2}$ can be computed as

$$
\mathrm{Cov}[X_1, X_2] = \mathrm{Cov}\left[\frac{F_{T_1}}{E_{g_1}}, \frac{F_{T_2}}{E_{g_2}}\right] = \sigma^2_{X_i}\rho_X \tag{16}
$$

where $\sigma^2_{X_i}$ is given by eq. (15), and the correlation coefficient, ρ_X, comes from the transformation (Fenton and Griffiths 2008)

$$
\rho_X = \frac{\exp\{\mathrm{Cov}[\ln X_1, \ln X_2]\} - 1}{\exp\left(\sigma^2_{\ln X_i}\right) - 1} \tag{17}
$$

and $\sigma^2_{\ln X_i}$ is defined by eq. (12). In addition

$$
\begin{aligned}
\mathrm{Cov}[\ln X_1, \ln X_2] &= \mathrm{Cov}[\ln F_{T_1} - \ln E_{g_1}, \ln F_{T_2} - \ln E_{g_2}] \\
&= \mathrm{Cov}[\ln F_{T_1}, \ln F_{T_2}] + \mathrm{Cov}[\ln E_{g_1}, \ln E_{g_2}] \\
&\approx \sigma^2_{\ln F_T}\rho_{\ln F_T} + \sigma^2_{\ln E}\gamma_{ff}
\end{aligned} \tag{18}
$$

again assuming F_T and E_g are independent. The correlation $\rho_{\ln F_T}$ is given by

$$
\rho_{\ln F_T} = \frac{\ln\left(1 + \rho_{F_T}v_T^2\right)}{\ln\left(1 + v_T^2\right)} = \frac{\ln\left(1 + \rho_{F_T}v_T^2\right)}{\sigma^2_{\ln F_T}} \tag{19}
$$

where ρ_{F_T} is the correlation between loads F_{T_1} and F_{T_2}. The term γ_{ff} is the average correlation coefficient between two log-elastic modulus fields of sizes $V_f = B \times B \times C$ surrounding the two piles, which are separated by distance s (see Fig. 2 for s). In detail, γ_{ff} is given by

$$
\gamma_{ff} = \frac{1}{V_f^2}\int_0^{V_f}\int_0^{V_f}\rho_{\ln E}\left(\sqrt{s^2 + (\underset{\sim}{x}_1 - \underset{\sim}{x}_2)}\right)^2 d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2 \tag{20}
$$

where $\rho_{\ln E}$ is given by eq. (4).

Employing eqs. (12) and (18) in eq. (17) leads to

$$
\begin{aligned}
\rho_X &= \frac{\exp\left(\sigma^2_{\ln F_T}\rho_{\ln F_T} + \sigma^2_{\ln E}\gamma_{ff}\right) - 1}{\exp\left(\sigma^2_{\ln F_T} + \sigma^2_{\ln E}\gamma_f\right) - 1} \\
&= \frac{\left(1 + \rho_{F_T}v_T^2\right)\left(1 + v_E^2\right)^{\gamma_{ff}} - 1}{\left(1 + v_T^2\right)\left(1 + v_E^2\right)^{\gamma_f} - 1}
\end{aligned} \tag{21}
$$

and using eqs. (15) and (21) in eq. (16) results in

$$
\mathrm{Cov}[X_1, X_2] = \frac{\mu_T^2}{\mu_E^2}(1 + v_E^2)^{\gamma_f+1}\left[\left(1 + \rho_{F_T}v_T^2\right)\left(1 + v_E^2\right)^{\gamma_{ff}} - 1\right] \tag{22}
$$

Finally, substituting eqs. (15) and (22) into eq. (11) gives

$$
\begin{aligned}
\mathrm{Var}\left[\frac{F_{T_1}}{E_{g_1}} - \frac{F_{T_2}}{E_{g_2}}\right] = 2\frac{\mu_T^2}{\mu_E^2}\left(1 + v_E^2\right)^{\gamma_f+1}\Big[&\left(1 + v_T^2\right)\left(1 + v_E^2\right)^{\gamma_f} \\
&- \left(1 + \rho_{F_T}v_T^2\right)\left(1 + v_E^2\right)^{\gamma_{ff}}\Big]
\end{aligned} \tag{23}
$$

from which the variance of differential settlement, σ^2_Δ, becomes

$$
\begin{aligned}
\sigma^2_\Delta = 2\delta^2_{\det}(1 - \eta)^2\left(1 + v_E^2\right)^{\gamma_f+1}\Big[&\left(1 + v_T^2\right)\left(1 + v_E^2\right)^{\gamma_f} \\
&- \left(1 + \rho_{F_T}v_T^2\right)\left(1 + v_E^2\right)^{\gamma_{ff}}\Big]
\end{aligned} \tag{24}
$$

If $\rho_{F_T} = 1$, so that the loads F_{T_1} and F_{T_2} are the same, then the above equation simplifies to

$$
\sigma^2_\Delta = 2\delta^2_{\det}(1 - \eta)^2\left(1 + v_E^2\right)^{\gamma_f+1}\left(1 + v_T^2\right)\left[\left(1 + v_E^2\right)^{\gamma_f} - \left(1 + v_E^2\right)^{\gamma_{ff}}\right] \tag{25}
$$

For $s \to \infty$, pile settlements δ_1 and δ_2 become independent, and hence both η and γ_{ff} in eq. (25) become zero. In this case, eq. (25) reduces to

$$
\sigma^2_\Delta = 2\delta^2_{\det}\left(1 + v_E^2\right)^{\gamma_f+1}\left(1 + v_T^2\right)\left[\left(1 + v_E^2\right)^{\gamma_f} - 1\right] \tag{26}
$$

Assuming that the normal distribution is a reasonable approximate distribution of the differential settlement between two piles, then the probability that the differential settlement exceeds $\Delta_{\max}$ is

$$
\begin{aligned}
p_f &= \mathrm{P}[|\Delta| > \Delta_{\max}] = \mathrm{P}[\Delta < -\Delta_{\max} \cup \Delta > \Delta_{\max}] \\
&= 2\mathrm{P}[\Delta < -\Delta_{\max}] = 2\Phi\left(\frac{-\Delta_{\max} - \mu_\Delta}{\sigma_\Delta}\right) \\
&= 2\Phi\left(\frac{-\Delta_{\max}}{\sigma_\Delta}\right) = \Phi(-\beta)
\end{aligned} \tag{27}
$$

because $\mu_\Delta = 0$, and where σ_Δ is calculated via eq. (24). The probability of failure, p_f, can also be expressed in terms of the reliability index, as shown in eq. (27), where Φ is the standard normal cumulative distribution function. That is, the reliability index corresponding to a particular value of p_f can be obtained by inverting eq. (27), $\beta = -\Phi^{-1}(p_f)$.

Validation of theory via Monte Carlo simulation

In this section, the predicted parameters of the distribution of differential settlement, Δ, are compared with Monte Carlo simulation results to assess the accuracy of the theory developed in the previous section. The particular case considered in this validation study is detailed in Table 1.

Realizations of differential settlement of two piles are obtained using the RFEM (Fenton and Griffiths 2008).

Three load cases (F_T) are considered for this analysis, as listed in Table 1, and it is assumed that the pile loads are equal and nonrandom. For each load case, a design pile length is determined as follows to achieve a target maximum total settlement, $\delta_{\max}$ = 0.025 m (Naghibi et al. 2014*a*):

$$
H = d\left[\left(\frac{1}{(\delta_{\max}\varphi_{gs}\mu_E d/F_T) - a_0}\right)^{1/a_2} - a_1\right] \tag{28}
$$

where φ_{gs} is the geotechnical resistance factor, accounting for uncertainty in geotechnical resistance. In this validation study,

Table 1. Input parameters used in the validation of theory.

Parameter	Values considered
d (m)	0.3
E_p (MPa)	21 000
F_T (MN)	1.46, 2.16, 3.16
μ_E (MPa)	30
v_E	0.1, 0.3, 0.5
Poisson's ratio, ν	0.3
$\theta_{\ln E}$ (m)	0.01, 0.1, 0.5, 1.0, 5.0, 10.0
s	$2d$, $3d$, $5d$, $10d$, $15d$, $20d$, $30d$
n_{sim}	2000
B (m)	2
C	$2H$

φ_{gs} is taken to be 1.0. The pile to soil stiffness ratio is assumed to be k = 700, from which a_0 = 0.029, a_1 = 2.44, and a_2 = 0.939 are obtained using the following regression developed by Naghibi et al. (2014*a*):

$$\begin{aligned} a_0 &= 2069.4633(k + 350)^{-1.6054} \\ a_1 &= 0.07 + (0.2934k^{0.3108}) \\ a_2 &= 0.6903 + (8.2464k^{-0.5268}) \end{aligned} \tag{29}$$

The resulting design pile lengths are H = 2, 4, and 8 m for the three load cases F_T = 1.46, 2.16, and 3.16 MN, respectively.

The mechanical interaction factor, η, is obtained for each pile length (load case) and each pile spacing, s, listed in Table 2, using Fig. 3.

The soil volume surrounding the pile, $V_f = B \times B \times C$, for use in the geometric average given by eq. (6), was selected by trial-and-error (see Naghibi et al. 2014*b*) and the (approximately) best averaging volume was found to occur when B = 2 m, and C = 2H. These choices led to the best agreement between theory and simulation with respect to settlement exceedance probabilities for a single pile.

Figures 4 and 5 illustrate the comparison between the theory and simulation-based estimates of $\mu_{|\Delta|}$ and σ_Δ. The theoretical estimates were obtained using eqs. (5) and (24) for $\rho_{F_T} = 1$ (identical loads) and $v_T = 0$ (nonrandom load). Figure 4 demonstrates that the theory underestimates $\mu_{|\Delta|}$ when $\mu_{|\Delta|}$ is small. Although not seen in the figure, the underestimation occurs most strongly for longer piles and smaller values of v_E and $\theta_{\ln E}$. The discrepancies between simulation and theory seen in Figs. 4 and 5 arise mainly because of approximations made in the theory. For instance, the theory assumes that Δ is normally distributed — this is not a good assumption when $\mu_{|\Delta|}$ is small, which is where the errors become more pronounced. Note that the errors are actually quite small in absolute value and thus not of great importance. For example, when the simulated $\mu_{|\Delta|}$ is less than 0.5 mm, the predicted $\mu_{|\Delta|}$ is often less than about 0.1 mm. In either case, the differential settlement is negligible.

The probability that the differential settlement exceeds Δ_{max}, as predicted by eq. (27), is compared with the simulation in Fig. 6 for three possible maximum acceptable differential settlement to pile spacing ratio values, Δ_{max}/s = 1/200, 1/500, and 1/1000. These serviceability gradient limitations are as specified in the *Canadian foundation engineering manual* (CFEM) (Canadian Geotechnical Society 2006).

It is evident from Fig. 6 that the theory sometimes significantly underestimates P[$|\Delta| > \Delta_{max}$], which is unconservative. Although not shown in Fig. 6, the disagreement is worst for smaller pile spacings, s. Note that the simulation involves fewer simplifications than does the theory, and so the simulation results are be-

Table 2. Input parameters used in Fig. 8.

Parameter	Values considered
d (m)	0.3
E_p (MPa)	21 000
μ_E (MPa)	30
v_E	0.1, 0.3, 0.5
φ_{gs}	1.0
$\theta_{\ln E} = s$	$2d$ to $67d$
B (m)	2
C	$2H$
Δ_{max}/s	1/200, 1/500, 1/1000
Target β	2.9

Fig. 4. Predicted, obtained via eq. (5), versus simulated mean absolute differential settlement, $\mu_{|\Delta|}$, for all cases listed in Table 1.

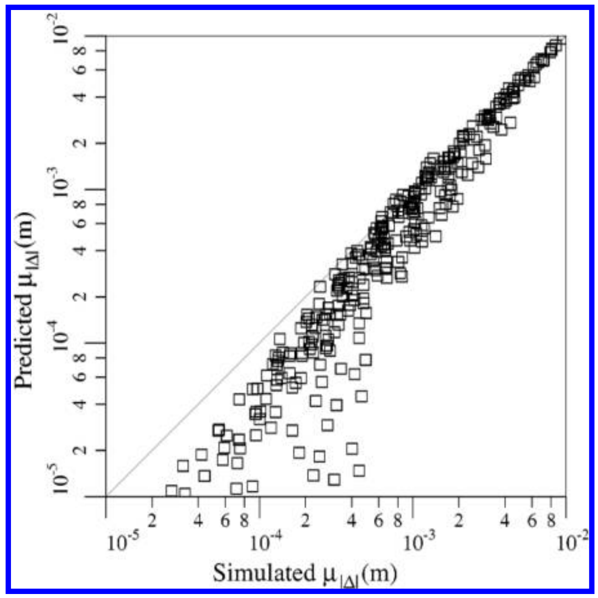


Fig. 5. Predicted, obtained via eq. (24), versus simulated standard deviation of differential settlement, σ_Δ, for all cases listed in Table 1.

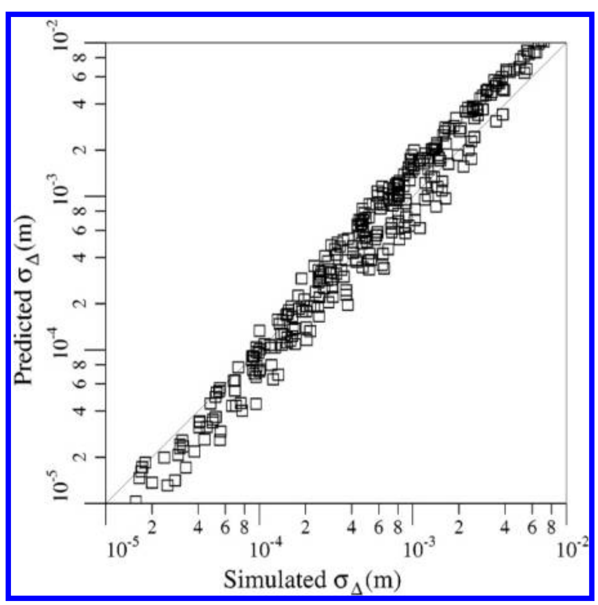
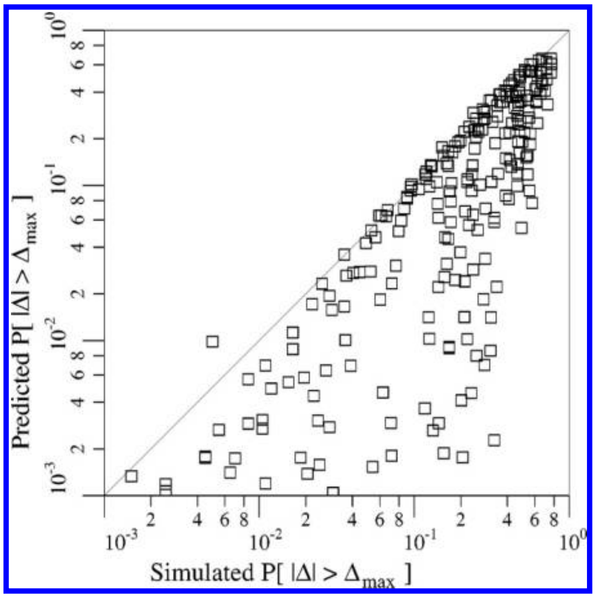


Fig. 6. Predicted, via eq. (27), versus simulated P[$|\Delta| > \Delta_{max}$] for all cases listed in Table 1.



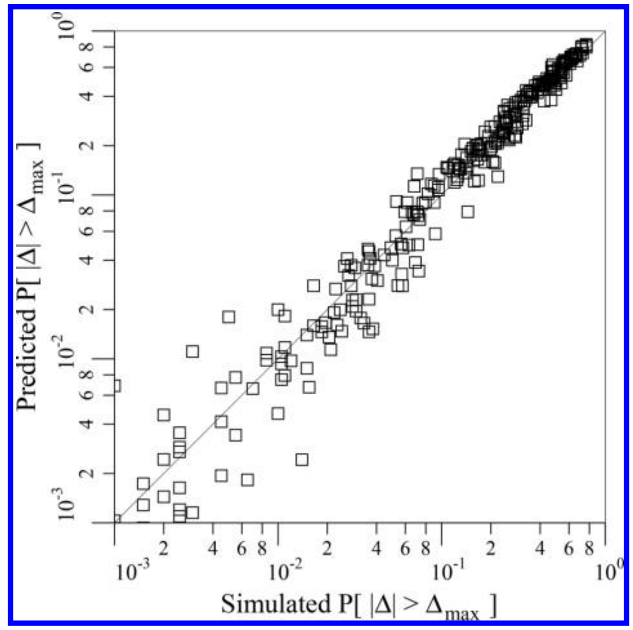
lieved to be more accurate, as was also shown by Naghibi et al. (2014*a*).

It is believed that the discrepancy between theory and simulation in Fig. 6 is due to the covariance between the piles being essentially overestimated in the theory by including both a statistical covariance component (γ_{ff}) at the same time as a mechanical interaction term (η). The overestimation in the "equivalent" covariance between the piles reduces the theoretically predicted mean differential settlement, as seen in Fig. 4, and thus significantly reduces the theoretical probability of excessive settlement, as seen in Fig. 6. This discrepancy can be largely solved by introducing an empirical correction to the theory, which was found by trial-and-error. If the value of η is replaced by -0.5η for short piles (e.g., $H < 3$ m), by 0 for medium length piles (e.g., $3 \le H \le 6$ m), and by 0.5η for longer piles (e.g., $H > 6$ m), then the agreement between theoretical and simulated exceedance probabilities is significantly improved, as shown in Fig. 7.

What this empirical adjustment is essentially doing is reducing the covariance between the piles — the amount of covariance reduction is greatest for shorter piles, where most of the errors were seen, as the statistical covariance portion is relatively higher for shorter piles. The application of an empirical correction raises the question as to why the statistical covariance is not reduced simply by reducing the volume $V_f = B \times B \times C$, rather than by reducing the mechanical component. The reason is that the choice in V_f led to a good prediction of the distribution of the settlement of an individual pile, and so it was felt that its size was an appropriate measure of the zone around the pile influencing the total settlement. The problem really is that the mechanical interaction factor was determined from a deterministic (nonrandom) elastic modulus field. The actual mechanical interaction factor in a spatially random elastic modulus field is unknown and not easy to specify probabilistically. Preliminary results using trials having identical random field realizations indicate that the random mechanical interaction factor is always lower than the deterministic one. However, it was felt that the determination of the distribution of the actual η was beyond the scope of this paper, and perhaps not worth the effort, as the empirical adjustment suggested above seems to work quite well.

With this empirical correction, the agreement between theory and simulation is considered very good. Thus, the theory is believed to be reliable enough to assist in design recommendations,

Fig. 7. Predicted, obtained via eq. (27), versus simulated P[$|\Delta| > \Delta_{max}$], adjusted by replacing η with -0.5η for $H < 3$ m, with zero for $3 \le H \le 6$ m, and with 0.5η for $H > 6$ m.



as will be discussed shortly. In other words, the normal distribution, along with an empirically adjusted standard deviation, can be used as a reasonable approximation to the distribution of the differential settlement between two piles.

Design recommendations

The objective of this section is to study how the design maximum tolerable total settlement, δ_{max}, for an individual pile, relates to the distribution of the differential settlement between two piles and, more specifically, to the maximum tolerable gradient, Δ_{max}/s.

The design δ_{max} required to achieve a certain reliability against excessive differential settlement can be obtained by considering eq. (28), which is re-written here as

$$(30) \qquad H = d\left[\left(\frac{1}{(q\delta_{max}\varphi_{gs}) - a_0}\right)^{1/a_2} - a_1\right]$$

in terms of q, having units of m^{-1}, which is defined by

$$(31) \qquad q = \mu_E d/\mu_T$$

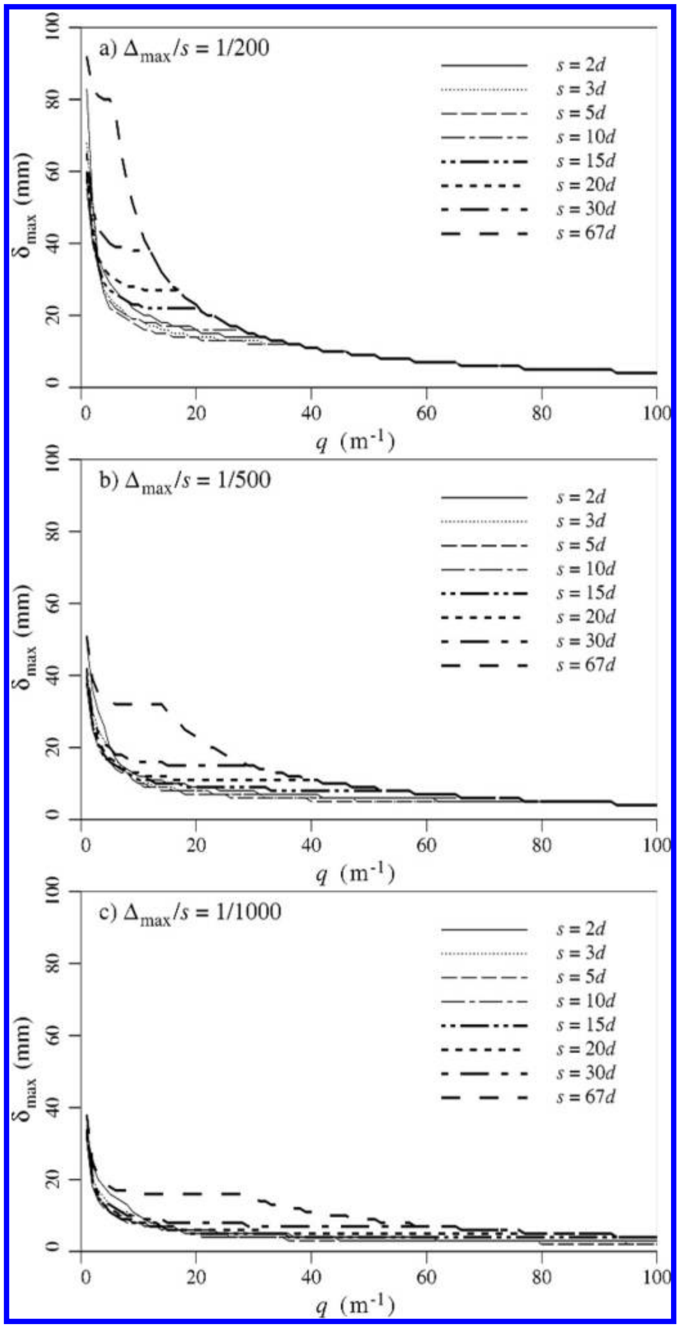
For given values of δ_{max}, q, and d, the design pile length, H, can be determined from eq. (30) ($k = E_p/\mu_E = 700$ is assumed here), which may be then used in eqs. (13) and (20) to find γ_f and γ_{ff}. The variance of Δ_{max} is then obtained from eq. (25), as follows:

$$(32) \qquad \sigma_\Delta^2 = 2\delta_{max}^2(1-\eta)^2(1+v_E^2)^{\gamma_f+1}(1+v_T^2)[(1+v_E^2)^{\gamma_f} - (1+v_E^2)^{\gamma_{ff}}]$$

where δ_{det} has been replaced by the target design total settlement, δ_{max}.

Based on results presented in Naghibi et al. (2014*b*), the correlation length, $\theta_{\ln E}$, leading to the highest mean differential settlement, is approximately equal to the distance between the two piles, s, and hence $\theta_{\ln E} = s$ is used for the estimation of σ_Δ in eq. (32). Once σ_Δ is known, the probability of excessive settlement can be determined by eq. (27). If the exceedance probability is larger than acceptable, then δ_{max} must be reduced. This entire

Fig. 8. Recommended design δ_{max} versus q for various pile spacings, s, and maximum acceptable gradient, Δ_{max}/s = (*a*) 1/200, (*b*) 1/500, and (*c*) 1/1000.



process can be repeated over a range of s, q, and δ_{max} values to determine the design δ_{max} value required to just achieve a target reliability, β.

The results of this iterative process are shown in Fig. 8 using the parameters specified in Table 2. Note that the results are not sensitive to the choices in pile diameter, d, and the mean values, μ_E and μ_T, because the design pile length takes all of these parameters into account — the actual settlement distribution will have a mean that shifts accordingly with the assumed mean in the elastic modulus field and the load, as well as with pile diameter. The probability of excessive settlement thus depends only on the variability of the ground, and the probability of excessive differential settlement depends on δ_{max}, pile spacing, pile length, and correlation length.

Figure 8 shows the required δ_{max} as a function of q for three maximum acceptable gradients — Δ_{max}/s = 1/200, 1/500, and 1/1000 (as specified by the CFEM, Canadian Geotechnical Society 2006) — and for the various pile spacings, s, listed in Table 2, to achieve a target reliability index β = 2.9. This target reliability corresponds to a maximum acceptable failure probability of about 1/500, which is what was assumed to be a typical maximum acceptable failure probability for SLS design in the Canadian highway bridge design code (Canadian Standards Association 2014). The plots in Fig. 8 can be used for design by drawing a vertical line at the specified q value and then reading off the required δ_{max} for a given pile spacing, s. For example, for applied load μ_T = 1.0 MN, soil elastic modulus μ_E = 20 MPa, and pile diameter d = 0.5 m, $q = 20 \times 0.5/1.0 = 10$ m^{-1} and pile spacing $s = 3d = 1.5$ m; δ_{max} values of 19, 12, and 8 mm are recommended for the three acceptable gradients: Δ_{max}/s = 1/200, 1/500, and 1/1000, respectively.

Note that the stepped nature appearing in Fig. 8 is due to the fact that discrete values of δ_{max}/s are considered in the iterative process described above, rather than continuous ones. Thus, each discrete value of δ_{max} applies to a continuous range in q.

The following observations can be made about the recommended δ_{max} values in Fig. 8:

1. By comparing Figs. 8*a*, 8*b*, and 8*c*, it is evident that the design value of δ_{max} increases with increasing Δ_{max}/s. This is as expected, because increasing Δ_{max}/s results in a less restrictive SLS requirement.
2. Within each plot in Fig. 8, it appears that the design value of δ_{max} increases with increasing s, for fixed Δ_{max}/s. Although this observation seems to be counter to the first (as s increases, it seems that Δ_{max}/s should decrease), it is important to remember that each plot in Fig. 8 is for a fixed Δ_{max}/s. Thus, an increase in s is accompanied by a corresponding increase in Δ_{max}, so that all curves in Fig. 8 have different Δ_{max} values. The net result is that when Δ_{max}/s is fixed, increasing s results in increasing δ_{max}. This is also as expected, because for a fixed maximum gradient, increased spacing between foundations results in a less restrictive SLS requirement.
3. The design δ_{max} decreases with increasing q. When q is large, the elastic modulus is large and the load is small, which leads to short piles. Short piles have larger variability in both their settlement and differential settlement. Thus, to achieve a certain reliability against excessive differential settlement, smaller values of δ_{max} are required when q is large.

Conclusions

The differential settlement between two piles has been studied and a theoretical model with an empirical adjustment has been developed, which was then validated by simulation. The theoretical model can be used to estimate the probability of excessive differential settlement and hence to provide design recommendations. The relationship between the target maximum total settlement recommended in design codes and the distribution of the differential settlement between two piles was investigated and design recommendations were made on how to select the target maximum total settlement of an individual pile to avoid excessive differential settlement between a pair of piles.

The differential settlement model presented here is a function of the load and ground stiffness distributions along with the distance, correlation coefficient (in both loads and ground parameters), and mechanical interaction between the piles. The local averages used around the piles gave very good agreement between predicted and simulated exceedance probabilities for total settlement in the study by Naghibi et al. (2014*b*). However, using the same local averages in this paper overemphasized the correlation between piles. To compensate, an empirical adjustment factor was introduced. The resulting probabilistic model is quite general and the agreement between the model and differential settlement simulation results was deemed to be very good (see Fig. 7).

One of the goals of this paper was to produce design recommendations that allow the designed pile system to avoid excessive differential settlement between pairs of piles. As the SLS design of foundations traditionally involves the specification of, and design against, a maximum total settlement, δ_{max}, it makes sense to continue that traditional design approach here. To this end, this paper provides maximum tolerable settlement values that correspond to a reliability index of at least 2.9 ($p_f = 1/500$) against excessive differential settlement. Note that the recommendations made here are based on variability in the ground and not on variability in the loads. That is, $v_T = 0$ was used in the design recommendations and the loads applied to the two piles were assumed equal. The actual joint load distribution is dependent on the stiffness of the supported structure, amongst other things, and the stiffness of the supported structure also influences the maximum allowable differential settlement. To properly model the joint load distribution, and the maximum tolerable differential settlement, a model of the supported structure would be required, which was beyond the scope of this paper. The assumptions made here essentially correspond to that of a very stiff supported structure (e.g., a pile cap). If the structure is actually quite flexible, one would expect increased differential settlement, but at the same time one would expect additional tolerance for differential movement. It is felt that the results presented here are basically applicable regardless of the structural model. Nevertheless, research is ongoing into the effect that structural stiffness and load-transfer mechanisms have on the SLS design of individual foundations against excessive differential movement.

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List of symbols

a_i settlement prediction parameter
B width of soil volume used in the geometric average around the pile
C depth of soil volume used in the geometric average around the pile
Cov[X,Y] covariance between random variables X and Y
d pile cross-sectional dimension (= 0.3 m)
E soil's elastic modulus (random)
E_g equivalent uniform soil elastic modulus that, if surrounding the pile, would yield the same settlement
E_p pile's elastic modulus
$E(\underset{\sim}{x}_i)$ elastic modulus at the spatial location $\underset{\sim}{x}_i$
E[.] expectation operator
F_T true total load (random)
H designed pile length
k pile to soil stiffness ratio (= E_p/μ_E)
n_{sim} number of simulations
P[.] probability operator
p pile perimeter
p_f probability of failure (=P[$|\Delta| > \Delta_{max}$])
q $\mu_E d/\mu_T$
s centre-to-centre pile spacing
V_f volume of geometric average around the pile (= $B \times B \times C$)
v_E elastic modulus coefficient of variation (= σ_E/μ_E)
v_T total load coefficient of variation (= σ_T/μ_T)
X_i ratio of load to elastic modulus geometric average (= F_{T_i}/E_{g_i})
$\underset{\sim}{x}$ spatial coordinate, (x, y, z) in 3-D
β reliability index (= $-\Phi^{-1}(p_f)$)
γ_f variance function giving variance reduction due to averaging over the soil volume surrounding the pile
γ_{ff} average correlation coefficient between elastic modulus along two piles over the volume V_f
Δ differential settlement between two piles
Δ_{max} maximum acceptable differential settlement
δ overall settlement of pile due to its loading and due to the settlement of a neighboring pile
δ' settlement of an individual pile
δ_{det} pile settlement when $E = \mu_E$ everywhere
δ_{max} maximum acceptable settlement for an individual pile
η mechanical interaction factor
$\theta_{\ln E}$ isotropic correlation length of the random log-elastic modulus field
μ_E mean elastic modulus
$\mu_{\ln E}$ mean of ln$E$
$\mu_{\ln E_g}$ mean of lnE_g
$\mu_{\ln F_T}$ mean of lnF_T
$\mu_{\ln X_i}$ mean of lnX_i
μ_{X_i} mean of X_i
μ_Δ mean differential settlement
$\mu_{|\Delta|}$ mean absolute differential settlement
μ_T mean total load
ν Poisson's ratio
v_E coefficient of variation of E
ρ_{F_T} correlation coefficient between the loads applied to the two piles
$\rho_{\ln E}$ correlation coefficient between ln E at two points
$\rho_{\ln F_T}$ correlation coefficient between the logarithm of the loads applied to the two piles
ρ_X correlation coefficient between X at the two piles ($X = F_T/E_g$)
σ_E standard deviation of elastic modulus
$\sigma_{\ln E}$ standard deviation of log-elastic modulus
$\sigma_{\ln E_g}$ standard deviation of lnE_g
$\sigma_{\ln F_T}$ standard deviation of the log-pile load
$\sigma_{\ln X_i}$ standard deviation of lnX_i
σ_{X_i} standard deviation of X_i
σ_Δ standard deviation of differential settlement
τ lag distance
Φ standard normal cumulative distribution function
φ_{gs} geotechnical resistance factor at SLS