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Abstract The random finite element method (RFEM) combines the random field theory and finite element method in the framework of Monte Carlo simulation. It has been applied to a wide range of geotechnical problems such as slope stability, bearing capacity and the consolidation of soft soils. When the RFEM was first developed, direct Monte Carlo simulation was used. If the probability of failure (p_f) is small, the direct Monte Carlo simulation requires a large number of simulations. Subset simulation is one of most efficient variance reduction techniques for the simulation of small p_f. It has been recently proposed to use subset simulation instead of direct Monte Carlo simulation in RFEM. It is noted, however, that subset simulation requires calculation of the factor of safety (FS), while direct Monte Carlo requires only the examination of failure or non-failure. The search for the FS in RFEM could be a tedious task. For example, the search for the FS of slope stability by the strength reduction method (SRM) usually requires much more computational time than a failure or non-failure checking. In this paper, the subset simulation is combined with RFEM, but the need for the search of FS is eliminated. The value of yield function in an elastoplastic finite element analysis is used to measure the safety margin instead of the FS. Numerical experiments show that the proposed approach gives the same level of accuracy as the traditional subset simulation based on FS, but the computational time is significantly reduced. Although only examples of slope stability are given, the proposed approach will generally work for other types of geotechnical applications.

Keywords Slope stability · Probability of failure · Random field · Finite element method

Introduction

By the very nature of their origins, geotechnical materials such as soils and rocks are variable in their engineering properties. This variability is rarely taken into account directly in traditional geotechnical analysis, rather some "representative average" or "appropriately conservative" property is usually assumed to act across the whole region of interest. The search for the safety of geotechnical projects is calculated using analytical methods and quantified by a safety factor to keep the p_f to an acceptable level. The FS itself can be overly conservative in some cases (Duncan 2000). It is common to use the same FS for different types of application without regard to the degree of uncertainty involved in its calculation. Through regulation or tradition, the same safety factor is often applied to conditions that involve widely varying degrees of uncertainty. This approach does not factor in the level of variability nor the consequences of failure. High probability events attracting a high FS may have negligible failure consequences and should be designed more economically. The deterministic approach also does not necessarily reflect the risk tolerance of the client or contractor and does not allow them to make informed decisions. This is not a very logical strategy (Duncan 2000) and adoption of probabilistic methods, where risk, defined as a function of probability and consequence, is explicitly quantified would be a better approach.

In the last two decades, the geotechnical community has begun to use probabilistic methods to take uncertainties directly into account (Fenton and Griffiths 2008). On the one hand, statistical methods and random field theory (Vanmarcke 1984) are used to characterize and model spatial variability explicitly (Jaksa et al. 2005; Phoon and Kulhawy 1999) and, on the other hand, various probabilistic methods ranging from simple methods such as first order second moment (FOSM) method, first order reliability method (FORM) to more advanced random finite element method (RFEM) (Griffiths and Fenton 2000; Griffiths and Fenton 2004) are used to predict the reliability of geotechnical structures. Huang et al. (2010b) showed, however, that simple methods (e.g. FOSM and FORM) can lead to unrealistic results because spatial variability cannot be properly modelled. Random field theory combined with FEM provides a systematic way to study the effects of spatial variability of soil properties on the reliability of geotechnical structures (Griffiths et al. 2009; Huang et al. 2010; Jiang et al. 2015; Jiang et al. 2014).

When RFEM was first developed, the direct Monte Carlo simulation was used. It is well known that direct Monte Carlo simulation becomes computationally time-consuming when estimating small p_f. The efficiency of Monte Carlo simulation can be enhanced by the application of variance reduction methods such as importance sampling (Au and Beck 2003) or subset simulation (Au and Beck 2001; Papaioannou et al. 2015). In problems with a large number of random variables, the Euclidean norm of the importance sampling weighting becomes unbounded so that the importance sampling is not applicable. Subset simulation turns out to be efficient in simulating small p_f even in high dimensions. Li et al. (2015) combined RFEM with subset simulation to simulate small p_f of slopes. However, the efficiency of subset simulation was not investigated. As will be shown later in this paper, if the search for FS is needed, subset simulation actually costs more computational time than direct Monte Carlo simulation for a given level of accuracy. This paper is specifically aimed at significantly improving the efficiency of estimating slope failure probabilities—especially where these probabilities are small—by avoiding having to estimate the slope's factor of safety for every realization. Estimating the factor of safety for a single realization involves repeating the entire non-linear finite element analysis for each of a sequence of strength reduction factors (SRF) until the largest SRF which just leads to slope failure is found. The computational overhead of having to repeat this sequence of finite element analyses for each realization within the subset simulation algorithm is enormous. The proposed method avoids having to search for the factor of safety, leading to a much more efficient approach to estimating the probability of slope failure. It is proposed that the value of yield function can be used as an indication of safety margin. Numerical results show that the computational time can be reduced by a factor of more than forty without losing accuracy.

In "Subset simulation for small probability of failure" section, direct Monte Carlo and subset simulation are compared and reviewed. It is shown that direct Monte Carlo simulation does not need to search for the FS in slope stability analysis while the

traditional subset simulation method does. In "Deterministic factor of safety analysis of slope by FEM" section, the strength reduction method (SRM) for obtaining the FS of slopes is reviewed. It is shown that the SRM requires more computational time than direct failure checking. In "Random finite element method by subset simulation without the search for FS" section, the RFEM is combined with subset simulation without computing the FS. Instead of using the FS as an indicator of safety margin, the total value of the yield function at all Gauss points is used as a measure of safety margin. Examples are given in "Examples" section. It is shown that subset simulation including the searching for FS actually requires more computational time than direct Monte Carlo simulation. If the searching for FS is avoided, the computational time can be reduced by a factor of more than 40 between the two subset simulation methods considered. Concluding remarks are provided in "Concluding remarks" section.

Subset simulation for small probability of failure

The p_f of a system is computed as:

$$p_f = P(y(x)\le 0) = \int_{y(x)\le 0} f_X(x)dx \tag{1}$$

where $f_X(x)$ is the joint probability density function (PDF) of the input variables X and y is the limit state function and $y(x) < 0$ means failure. Limit states could relate to strength failure, serviceability failure, or anything else that describes unsatisfactory performance. If the FS is used to quantify the safety margin, the limit state function can be written as:

$$y(x) = FS-1 \tag{2}$$

Direct integration of Eq. (1) is usually impossible since many geotechnical problems do not have exact analytical solutions to the deterministic problem (e.g. slope stability problem). Monte Carlo simulation is commonly used as:

$$p_f \approx \frac{1}{N_{\text{sim}}}\sum_{i=1}^{N_{\text{sim}}} I_F(\mathbf{x}_i) \tag{3}$$

where N_{sim} is the number of simulations, $\mathbf{x}_i$ is the ith sample of X, and

$$I_F(\mathbf{x}_i) = \begin{cases} 1, & y(\mathbf{x}_i)\le 0 \\ 0, & y(\mathbf{x}_i) > 0 \end{cases} \tag{4}$$

Eq. (4) requires a deterministic analysis to check whether the sample fails or not. It should be noted, however, that this does not necessarily mean that the search for FS is needed. For example, in slope stability analysis, failure can be checked without using SRM, but by merely checking to see if the current realization fails to converge, implying stability failure.

The direct Monte Carlo simulation is robust and its efficiency does not depend on the dimension of X. Because I_F is a Bernoulli random variable, the coefficient of variation of the estimate of p_f is (for small p_f)

$$V_{p_f} \approx \sqrt{\frac{(1-p_f)}{p_f N_{\text{sim}}}} \tag{5}$$

It can be seen from Eq. (5) that direct Monte Carlo simulation requires a large number of simulations when p_f is very small. The subset simulation algorithm (Au and Beck 2001) provides an efficient algorithm for the estimation of small p_f. The basic idea of subset simulation is to express the p_f as a product of a sequence of conditional probabilities:

$$p_f = P_1 \prod_{i=1}^{m-1} P(F_{i+1}|F_i) \tag{6}$$

This can be achieved by defining a decreasing sequence of failure events. For example, if failure of a system is defined as $FS\le 1$, one can define a decreasing sequence of intermediate thresholds $FS_1 > FS_2 > \ldots > FS_m = 1$, such that:

$$p_f = P(FS\le 1) = P(FS\le FS_1)\prod_{i=1}^{m-1} P(FS\le FS_{i+1}|FS\le FS_i) \tag{7}$$

where $FS_m = 1$.

It can be seen that the intermediate conditional probability $P(FS\le FS_{i+1}|FS\le FS_i)$ is larger than $P(FS\le 1)$ and can be estimated by a smaller number of simulations, denoted as N. It is not clear beforehand how to choose the sequence of intermediate thresholds. Another option is to set the conditional probability $P(FS\le FS_{i+1}|FS\le FS_i)$ to a fixed value P_0, and the intermediate thresholds can be chosen as the $(1-P_0)N$ largest values among the N number of simulated FS. The simulation stops when $P(FS\le 1|FS\le FS_{m-1})\ge P_0$. The P_0 is chosen so that $N = 1/P_0$ is an integer.

The question left is how to compute the conditional probability $P(FS\le FS_{i+1}|FS\le FS_i)$. Au and Beck (2001) used the modified Metropolis algorithm. The modified Metropolis algorithm is based on a component-wise sample generation to avoid the small acceptance rate of the original Metropolis algorithm sampler in high dimensions. The method requires that the random variable space be independent. Fortunately, random fields are usually generated in independent standard normal space first and then transferred to real space using an appropriate algorithm. This means that RFEM can be driven in standard normal space by subset simulation.

The subset simulation based on the modified Metropolis algorithm can be summarized as:

1. $m = 1$, conduct N RFEM simulations. For each sample, evaluate the FS and sort them in ascending order $FS_1 < FS_2 < \cdots FS_{NP_0+1} < \cdots < FS_N$.
2. Set $m = m + 1$. For each of the samples with $FS < FS_{NP_0+1}$, generate $N_i = 1/P_0$ new samples according to the modified Metropolis algorithm:

   (a) For each component x_k, $k = 1, \cdots, d$, where d is the dimension of X, generate candidate $x_{k,c}$ using a symmetric proposal distribution $q(\cdot|x_{k,i-1})$ centred at $x_{k,i-1}$, where subscript $i-1$ refers to current sample. The standard normal distribution is usually used for the proposal distribution.

   Take

$$x_{k,i} = \begin{cases} x_{k,c} & \text{with probability } \alpha \\ x_{k,i-1} & \text{with probability } 1-\alpha \end{cases}$$

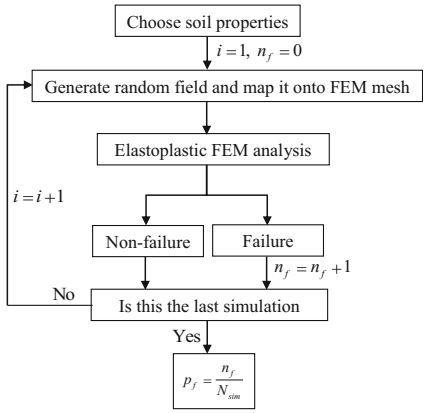

Fig. 1 RFEM and direct Monte Carlo simulation

where subscript i refers to the next sample, and

$$\alpha = \min\left\{1, \frac{p(x_{k,i-1})}{p(x_{k,c})}\right\}$$

and p is the probability density function of the target distribution, which in this present study, is the Standard Normal Distribution for the generation of random fields.

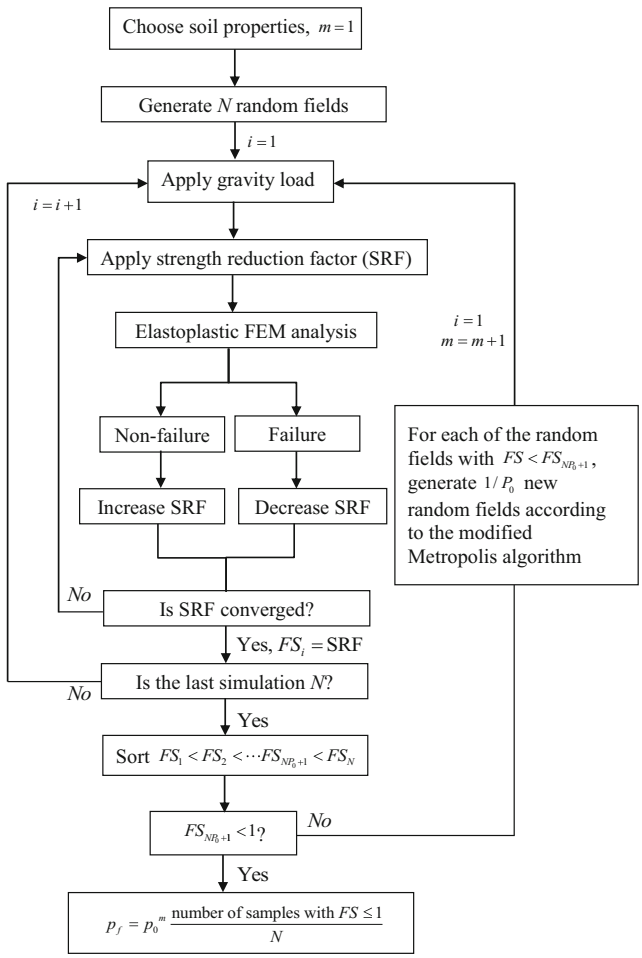

Fig. 2 Flow chart of RFEM by subset simulation with the search for FS

(b) For each new sample, generate a random field, map it onto the finite element mesh, and evaluate the *FS*. If $FS < FS_{NP_0+1}$, accept the new sample, otherwise keep the current sample.

3, Sort the newly obtained *FS* in ascending order. If $FS_{NP_0+1} < 1$, $p_f = p_0^m \frac{\text{number of samples with } FS\le 1}{N}$ and stop, otherwise go to 2.

If the proposal distribution is not centred at the current sample, or it is unsymmetric, the acceptance criterion needs to be adjusted as:

$$\alpha = \min\left\{1, \frac{p(x_{k,i-1})\,q\left(x_{k,i-1}\middle|x_{k,c}\right)}{p(x_{k,c})\,q\left(x_{k,c}\middle|x_{k,i-1}\right)}\right\}$$

This is called Metropolis-Hasting algorithm (Hastings 1970). The common choices of proposal distribution are the uniform and normal distributions centred at the current sample. The optimal intermediate probability of failure P_0 lies in the interval 0.1~0.3 (Au et al. 2007).

Although subset simulation is efficient in simulating small p_f, it requires the evaluation of limit state function so that the intermediate p_f can be calculated. In slope stability analysis, this is usually done by searching for the FS by the SRM. The SRM takes much more computational time than direct failure checking. If the search for FS can be avoided in subset simulation, the efficiency of subset simulation can be significantly improved.

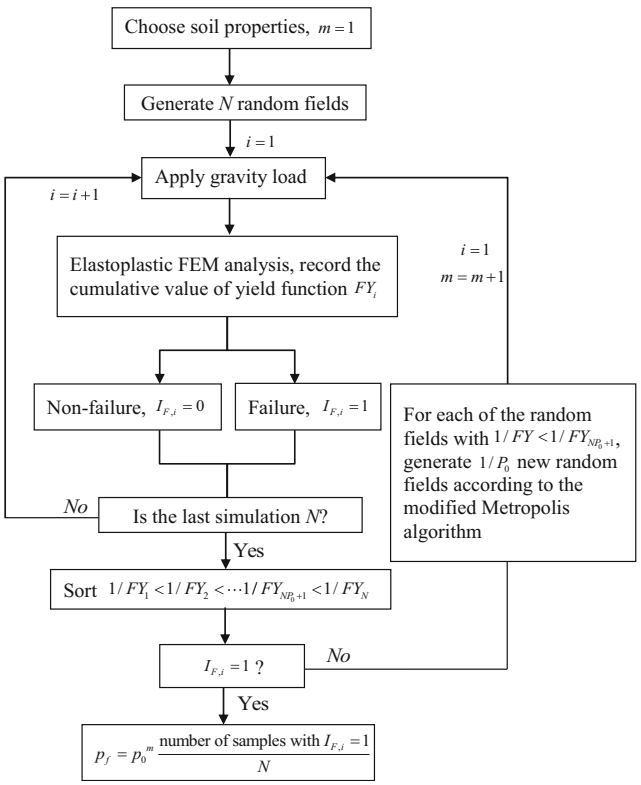

Fig. 3 Flow chart of RFEM by subset simulation without the search for FS

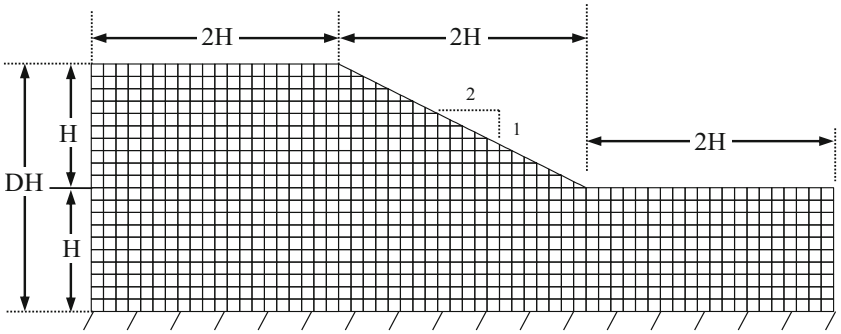

Fig. 4 An undrained slope

Deterministic factor of safety analysis of slope by FEM

Limit equilibrium method (LEM) and finite element method (FEM) are commonly used to estimate the FS of slopes. Compared to the LEM, the FEM makes no assumption in advance about the shape or location of the yield surface. Failure occurs naturally through the zones within the soil mass in which the soil shear strength is unable to sustain the applied shear stresses (Griffiths and Lane 1999). This feature is especially important when the strengths of soils vary spatially.

The starting point of elastoplastic FEM analysis is that the strength of soils is governed by a field criterion (all called yield function in this paper). The Mohr-Coulomb criterion remains the one most widely used in geotechnical practice. In terms of principal stresses and assuming a compression negative sign convention, Mohr-Coulomb criterion may be written as:

$$f = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}) + \frac{1}{2}(\sigma_{\max} + \sigma_{\min})\sin\phi' - c'\cos\phi' \tag{8}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum (least compressive) and the minimum (most compressive) principal stresses, respectively, and ϕ' and c' are the friction angle and cohesion of the soil.

In elastoplastic FEM analysis, the strain rate $\dot{\varepsilon}$ is split into an elastic component $\dot{\varepsilon}^e$ and a plastic component $\dot{\varepsilon}^p$

$$\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p \tag{9}$$

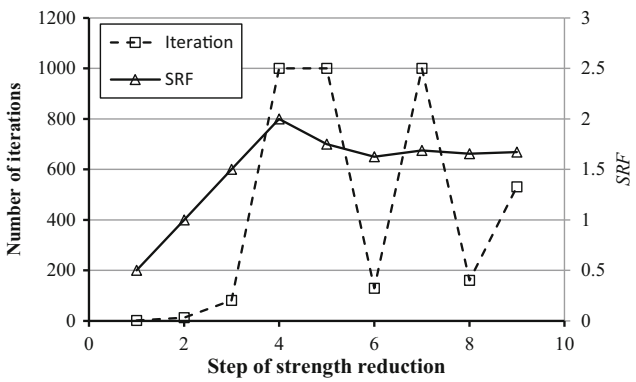

Fig. 5 The search for FS

Based on this decomposition, the elastic stress–strain relationship can be rewritten as:

$$\dot{\sigma} = \mathbf{D}^e\left(\dot{\varepsilon} - \dot{\varepsilon}^p\right) \tag{10}$$

where $\mathbf{D}^e$ is the elastic stress–strain tensor.Plastic strain rates for non-associated plasticity are assumed to follow the relations:

$$\dot{\varepsilon}^p = \dot{\lambda}\mathbf{q} \text{ and } \mathbf{q} = \frac{\partial g}{\partial \sigma} \tag{11}$$

where g is the plastic potential function, $\lambda \geq 0$ is the consistency parameter which represents the magnitude of the plastic flow and $\mathbf{q}$ is the flow direction given by the derivatives of the plastic potential function g with respect to stress. It is noted that the plastic strain-increments are associated with vectors perpendicular to the plastic potential surface.

In the FEM analysis of slope stability, the soil is initially assumed to be elastic and the model generates normal and shear stresses at all Gauss points within the mesh. These stresses are then compared with the yield criterion. If the stresses at a particular Gauss point lie within the yield envelope, then that location is assumed to remain elastic. If the stresses lie on or outside the yield surface, then that location is assumed to be yielding. When the stresses lie outside the yield surface, they have to be returned to the yield surface. Returning stresses to the yield surface is usually called the numerical integration of the constitutive equations (Huang and Griffiths 2008; Huang and Griffiths 2009). Overall, shear failure occurs when a sufficient number of Gauss points have yielded to allow a mechanism to develop. In this case, the algorithm is unable to converge within a user-defined iteration ceiling, the implication being that no stress distribution can be found that is simultaneously able to satisfy both the yield criterion and global equilibrium. If the algorithm is unable to satisfy these criteria, "failure" is said to have occurred.

It is noted that the above FEM analysis checks only if the slope fails or not. The safety margin is yet to be found. In deterministic approaches, the safety margin is usually quantified by the FS. The FS is the number by which the original shear strength parameters must be divided in order to bring the slope to the point of failure. For a detailed description of finite element slope stability analysis, the reader is referred to Griffiths and Lane (1999).

$$c_f = \frac{c}{\text{FS}} \text{ and } \phi_f = \arctan\left(\frac{\tan\phi}{\text{FS}}\right) \tag{12}$$

Table 1 Three undrained slopes

	μ_{c_u} (kPa)	V_{c_u}	θ (m)
Slope 1	57.82	0.2	4000
Slope 2	50.00	0.3	10
Slope 3	50.00	0.3	5

To obtain the FS, it is necessary to initiate a systematic search for the value of SRF that will just cause the slope to fail at which point FS = SRF. This is usually done by the bisection method, which costs more computational time than just checking failure.

Random finite element method by subset simulation without the search for FS

The RFEM is currently the most advanced probabilistic methods for slope stability. It involves the generation and mapping of a random field of properties onto a finite element mesh. Full account is taken of local averaging and variance reduction (Fenton and Vanmarcke 1990) over each element, and an exponentially decaying spatial correlation function is incorporated. The random field is initially generated and properties assigned to the elements. After application of gravity loads, if the algorithm is unable to converge within a user-defined iteration ceiling, failure is said to have occurred. The analysis is repeated numerous times using Monte Carlo simulations. Each realization of the Monte Carlo process involves the same mean, standard deviation and spatial correlation length of soil properties. However, the spatial distribution of properties varies from one realization to the next. Following a "sufficient" number of realizations, the p_f can be easily estimated by dividing the number of failures by the total number of simulations. Details of RFEM can be found in Griffiths and Fenton (2000) and Griffiths and Fenton (2004). Figure 1 shows the flow chart of RFEM by direct Monte Carlo simulation. It is noted that no searching for FS is needed.

It is well known that direct Monte Carlo simulation becomes computationally expensive when estimating small p_f. Since the random field in RFEM is generated in the underlying normal space, it is easy to use subset simulation. Li et al. (2015)) combined RFEM with subset simulation to estimate small p_f of slopes. It was shown that subset simulation takes fewer simulations than direct Monte Carlo simulation for a given level of accuracy. The general procedure of RFEM by subset simulation with the search for FS is summarized in Fig. 2.

In subset simulation, it is necessary to analyze the safety margin of each simulation so that the failure region in the probability space can be probed efficiently, but the price of searching for safety margins can be very high. This is especially true when the SRM is used to search for the FS. The resulting overall computational time of subset RFEM simulation may be more than direct Monte Carlo RFEM simulation. It is thus desirable to avoid the search for FS. It is noted that only the relative safety margin is needed in subset simulation. In a regular elastoplastic FEM analysis, the value of the yield function Eq. (8), denoted as *FY* hereafter, is an indicator of the safety margin. It can be used to compare the relative safety margins of all simulations from a set of subset simulation. An advantage of FS over FY is that FS is also an indicator of failure. If FS < 1, it means failure has occurred. However, although FY can be used as an indicator of safety margin, it cannot be used as an indicator of overall failure. Fortunately, the regular elastoplastic FEM analysis reveals if the slope fails or not. As mentioned previously, if the algorithm is unable to converge within a user-defined iteration ceiling, failure is said to have occurred. The indicator of failure, denoted as I_F, is set to one for failure and zero for non-failure in this paper. In summary, the search for the FS by the SRM can be replaced by recording the cumulative value of the yield function FY and the failure indicator I_F. As will be shown, it was found that the cumulative value of the yield function of all Gauss points from all elastoplastic iterations as a replacement for FS worked very well. The flow chart of RFEM by subset simulation without the search for FS is summarized in Fig. 3.

Examples

An 2:1 (angle 26.6°) undrained ($\phi_u = 0$) slope is considered with the slope profile shown in Fig. 4. The slope has height $H = 10.0$ m with depth ratio $D = 2$, soil unit weight γ_{sat}(or γ) = 20.0 kN/m^3, deterministic shear strength $c_u = 57.82$ kPa. Using the SRM with an iteration ceiling of 1000, the *FS* of the slope was found to be 1.69. The search for *FS* is shown in Fig. 5. It can be seen from Fig. 5 that there are totally 9 steps of strength reduction. At step 2 when SRF = 1, only 21 elastoplastic iterations were needed to reduce unbalanced loads to be significantly smaller than the applied loads. If the strength reduction is conducted to search for FS, the total number of elastoplastic iterations over 9 steps

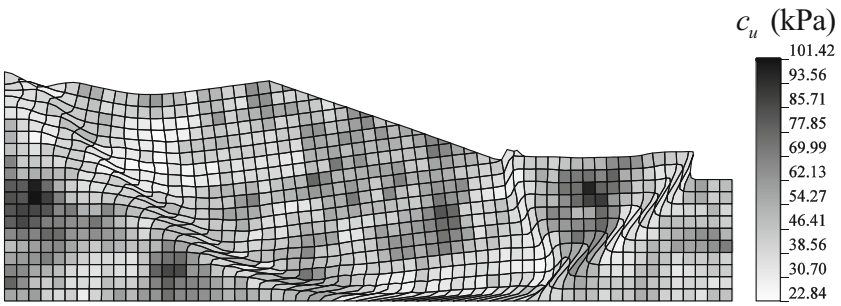

Fig. 6 A typical simulation with $\theta = 10$m

Table 2 Number of simulations

Slope	Direct Monte Carlo	Subset simulation with FS evaluation	Subset simulation without FS evaluation
1	10,000	6000	6000
2	100,000	6000	6000
3	100,000	10,000	10,000

of strength reduction is 3917. This means that the search for FS is roughly 190 times more expensive than simply checking for failure.

Because the FS is linearly proportional to the undrained shear strength for homogeneous undrained slopes, the undrained shear strength corresponding to FS = 1 is $c_{u,FS=1} = 34.21$ kPa.

If the undrained shear strength c_u is assumed to be a lognormally distributed random variable with mean $\mu_{c_u} = 57.82$kPa and coefficient of variation $V_{c_u} = 0.2$, the corresponding underlying normal mean and standard deviation are

$$\sigma_{\ln c_u} = \sqrt{\ln\left\{1 + V_{c_u}^2\right\}} = 0.20 \tag{13}$$

$$\mu_{\ln c_u} = \ln\mu_{c_u} - \frac{1}{2}\sigma_{\ln c_u}^2 = 0.50 \tag{14}$$

There is only one random variable, so p_f is simply equal to the probability that the shear strength parameter c_u will be less than $c_{u,FS=1} = 34.21$. The p_f is therefore given by:

$$p_f = p[c_u < 34.21] = \Phi\left(\frac{\ln 34.21 - \mu_{\ln c_u}}{\sigma_{\ln c_u}}\right) = 0.0055 \tag{15}$$

The undrained shear strength c_u is assumed to be a lognormally distributed random field with spatial correlation length θ of 4000 m. The exponential correlation function is used. The p_f estimated by RFEM with 10000 direct Monte Carlo simulation is 0.0059, which is close to the prediction using Eq. (15).

To investigate efficiency of subset simulations, another two slopes with different random fields are considered. All three slopes have the same geometry, and input parameters for slope 3 were changed so that the probability of failure is small. Because the proposed method avoids the need to search for the factor of safety during subset simulations, one example would be enough to show its efficiency as long as the probability of failure is small. The parameters are summarized in Table 1. Based on the mean strength, the FS of slopes 2 and 3 are found to be 1.47. There are 910 random variables in each random field. A typical simulation for $\theta = 10$ m is shown in Fig. 6.

To compare the efficiency of direct Monte Carlo simulation, subset simulation with FS evaluation and the subset simulation without FS evaluation, 50 independent simulation runs are computed for each. In all subset simulations, the P_o is set to 0.1 and 2000 simulations were conducted for each level of subset simulations. The standard normal distribution centred at the current sample is chosen as the proposal distribution. The number of simulations is shown in Table 2. The averaged p_f over 50 runs and the corresponding coefficient of variation of p_f $\left(V_{p_f}\right)$ are shown in Tables 3 and 4. The probability of failure of slope 2 is much higher than that of slope 3. This is because the spatial correlation length is 10 and 5 m for slopes 2 and 3, respectively. It is usually observed that the probability of failure of slope increases when the spatial correlation length increases. The sharpest increase happens when the spatial correlation length is approximately equal to the slope height, which in this case is 10 m. It can be seen from Table 3 that all three methods give similar estimates of p_f. This means that avoiding the search for FS did not affect the accuracy. It is noted that the V_{p_f} of slope 2 estimated by direct Monte Carlo simulation is significantly smaller than the other two slopes because a relatively large number of simulations were conducted. Compared to slope 1, ten times more simulations were conducted for slope 2, so the computational time is also approximately 10 times higher. The V_{p_f} estimated by direct Monte Carlo simulation in Table 4 is in good agreement with Eq. (5). The V_{p_f} estimated by subset simulation depends on the correlation between the conditional p_f estimated by each subset simulation. The upper bound estimate of V_{p_f} is obtained by assuming the conditional p_f is fully correlated:

$$V_{p_f} \approx \sqrt{\sum_{i,j=1}^{m} V_{p_{f,i}} V_{p_{f,j}}} \tag{16}$$

The coefficient of variation of p_f of the first level subset simulation $V_{p_{f,1}}$ can be estimated by Eq. (5). The coefficient of variation of p_f of the following levels of subset simulation can be estimated by:

$$V_{p_f,\, i>=2} \approx \sqrt{\frac{\left(1-p_{f,i}\right)}{p_{f,i}N}(1+\gamma)} \tag{17}$$

Table 3 Averaged p_f over 50 independent runs

Slope	Direct Monte Carlo	Subset simulation with FS evaluation	Subset simulation without FS evaluation
1	0.0059	0.0056	0.0062
2	0.0056	0.0066	0.0058
3	8.8×10^{-5}	10.5×10^{-5}	9.8×10^{-5}

Table 4 Coefficient of variation of p_f (V_{p_f}) over 50 independent runs

Slope	Direct Monte Carlo		Subset simulation with FS evaluation			Subset simulation without FS evaluation		
	Eq. (3)	Simulations	Lower bound	Upper bound	Simulations	Lower bound	Upper bound	Simulations
1	0.13							
	0.13	0.15	0.24	0.16	0.15	0.24	0.15	
2	0.042	0.037	0.15	0.24	0.17	0.15	0.24	0.17
3	0.34	0.34	0.24	0.49	0.35	0.24	0.49	0.34

where γ is related to the correlation among the samples within each subset simulation and $\gamma \leq 3$ (e.g. (Au and Beck 2001)).The lower bound estimate of V_{p_f} is obtained by assuming that the conditional p_f is uncorrelated:

$$V_{p_f} \approx \sqrt{\sum_{i}^{m} V_{p_{f,i}}^2} \tag{18}$$

Assuming $\gamma = 3$, the above estimations of the bounds of V_{p_f} are shown in Table 4. The V_{p_f} estimated by 50 independent subset simulation runs is close to the lower bound for slopes one and two, but is close to the average of the upper and lower bounds for slope 3. For all three slopes, when the search for FS is avoided, the estimated V_{p_f} are very similar to the ones estimated by searching for the FS. This confirms again that avoiding the search for FS did not affect the accuracy.

The computational time of the three methods are compared in Table 5. All simulations were conducted in a server with Intel Xeon CPU X5675@3.07 GHz with 128GB RAM. Several observations can be made from Table 5.

1. For direct Monte Carlo simulation, the averaged computational time per simulation decreases as the p_f decreases. This is because there are fewer failures when p_f is small, and the simulations that do not fail require fewer elastoplastic iterations than failed simulations.
2. For subset simulation, the averaged computational time per simulation increases as the p_f decreases. This is because subset simulation probes the failure region efficiently and thus has more failures on average.
3. A counter-intuitive observation is that subset simulations with the search for FS costs more computational time than direct Monte Carlo simulation for all three slopes considered. The smaller the p_f, the less efficient the subset simulation is. This means that subset simulations are inefficient in simulating small p_f if the search for FS is needed.
4. The efficiency of subset simulation can be significantly improved by avoiding the search for FS. This is especially true when p_f is small. For slope 3, the computational time is reduced by around 40 times using the proposed method. The computational time of each level of subset simulations is compared in Fig. 7. It can be seen from Fig. 7 that most of the computational time (above 80 %) are spent on the search for FS in the traditional subset simulations.
5. Subset simulation is not necessary more efficient than direct Monte Carlo simulation even when the search for FS is avoided. For the examples considered in this paper, the improved subset simulation is less efficient than direct Monte Carlo simulation when p_f is around 0.005 (slopes 1 and 2). For slope 3, when p_f is around 0.0001, subset simulation is about 3 times more efficient than direct Monte Carlo simulation.

The efficiency of the proposed method in the simulation of small probability of failure of slopes has been demonstrated by three slopes with same geometry but different parameters. A more generalized slope model with several soil layers and heterogeneous soil properties would be interesting, but it would not add anything new to demonstrate the efficiency of proposed method. The current subset simulation for slope stability problems requires the search for a factor of safety, which is very time-consuming. This paper developed a new method which avoids the search for factor of safety. The efficiency gained can be demonstrated clearly with the example used in the paper because the time taken to search for the factor of safety does not depend on the number of random numbers.

Concluding remarks

In this paper, the RFEM is combined with subset simulation and it is demonstrated that subset simulation is less efficient than direct Monte Carlo simulation when the search for FS is needed. It is proposed that the value of yield function can be used as a measure of safety margin instead of the FS in subset simulation. Numerical examples show that the efficiency of subset simulation can be significantly improved by avoiding the search for FS without losing accuracy.

Although only relatively simple slope stability examples are used to demonstrate the efficiency of the proposed method, it is applicable to any slope stability problem with any assumed random field models for the soil properties.

Table 5 Comparison of computational time (h)

Slope	Direct Monte Carlo	Subset simulation with FS evaluation	Subset simulation without FS evaluation
1	0.77	75.82	4.74
2	8.36	77.70	3.50
3	7.62	116.74	2.73

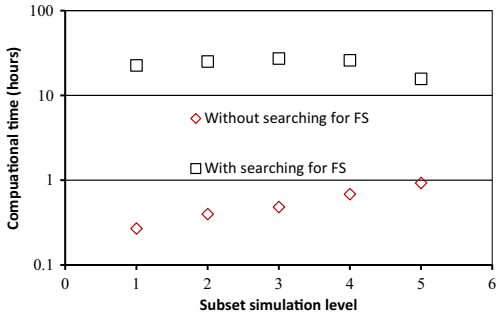

Fig. 7 Comparison of computational time of subset simulations

c_u undrained cohesion, c_f reduced undrained cohesion, $c_{u,FS=1}$ undrained cohesion when $FS = 1.0$, d dimension of X, D foundation depth ratio, $\mathbf{D}^e$ elastic stress–strain tensor, g plastic potential function, f yield function, f_X joint probability density function, FS factor of safety, FY cumulative value of yield function of all Gaussian points, H slope height, I_F indicator function of failure, k number of simulations of each level of subset simulation, m level of subset simulation, n number of random variables, n_f number of failures, N number of simulations of each level of subset simulation, N_{sim} number of simulations of direct Monte-Carlo simulation, p the Standard Normal Distribution, P_o fixed intermediate conditional probability of failure, p_f probability of failure, $\mathbf{q}$ flow direction, $q(\cdot|x_{k,i-1})$ proposal distribution, which should be symmetric and centred at current sample, SRF strength reduction factor, X random variables, $\mathbf{x}$ realization of random variables, $x_{k,c}$ proposed kth component of $\mathbf{x}$, $x_{k,i-1}$ current kth component of $\mathbf{x}$, $x_{k,i}$ new kth component of $\mathbf{x}$, y limit state function, α acceptance criterion, γ_{sat} saturated soil unit weight, γ a parameter that is related to the correlation among the samples within each subset simulation, θ spatial correlation length, μ_{c_u} mean undrained cohesion, V_{c_u} coefficient of variation of undrained cohesion, V_{p_f} coefficient of variation of probability of failure, ε strain, $\dot{\varepsilon}$ strain rate, ε^e elastic component of strain rate, ε^p plastic component of strain rate, λ consistency parameter, $\dot{\sigma}$ stress rate, $\sigma_{\max}$ maximum principal stress, $\sigma_{\min}$ minimum principal stress, $\sigma_{\ln c_u}$ equivalent normal standard deviation of undrained cohesion, ϕ_u undrained friction angle, ϕ_f reduced friction angle

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