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ARTICLE

# Target geotechnical reliability for redundant foundation systems

Farzaneh Naghibi and Gordon A. Fenton

**Abstract:** Geotechnical support systems (e.g., deep and shallow foundations) generally involve at least some redundancy. For example, if a building is supported by $n_p$ separate foundations, then failure (e.g., excessive settlement) of a single foundation will generally not result in failure of the building if the building is able to shed the load from the failed foundation to adjacent foundations. This load-shedding ability lends the foundation system redundancy — system failure only occurs if multiple foundations fail. This paper investigates the relationship between the level of geotechnical redundancy, individual foundation reliability, and system reliability for deep foundations (piles). In the particular case where the pile resistance remains constant after achieving its ultimate capacity (at a certain displacement), the relationship between individual and system reliabilities is computed theoretically. The more general case, where the load carried by the pile reduces after exceeding its ultimate capacity, is investigated by Monte Carlo simulation. Charts relating system and individual reliability indices are presented, which can be used to aid in the design of individual piles as part of a pile support system.

*Key words:* reliability-based design, deep foundation design, redundant systems, probabilistic modeling.

**Résumé :** Les systèmes de soutien géotechnique (par ex., des fondations superficielles et profondes) comportent en général au moins une certaine redondance. Par exemple, si un bâtiment est soutenu par $n_p$ des fondations, alors la défaillance (p. ex., le tassement excessif) d'une seule fondation n'entraînera généralement pas la défaillance de l'immeuble, si l'immeuble est en mesure de délester la charge de la fondation échouée aux fondations adjacentes. Cette capacité de délestage de la fondation prête la redondance au système de fondation — la défaillance du système se produit uniquement si plusieurs fondations échouent. Cet article étudie la relation entre le niveau de redondance géotechnique, la fiabilité de fondation individuelle et la fiabilité du système de fondations profondes (pieux). Dans le cas particulier où la résistance de pieu demeure constante après la réalisation de sa capacité ultime (à un certain déplacement), la relation entre la fiabilité individu et la fiabilité du système est calculée théoriquement. Le cas plus général, où la charge transmise par le pieu diminue après avoir dépassé sa capacité ultime, est étudié par simulation de Monte-Carlo. Des tableaux montrant des indices de fiabilité de systèmes et individus sont présentés, qui peuvent être utilisés pour faciliter la conception de pieux individuels dans le cadre d'un système de soutien de pieu. [Traduit par la Rédaction]

*Mots-clés :* conception basée sur la fiabilité, conception des fondations profondes, systèmes redondants, modélisation probabiliste.



## Introduction

Foundation systems are usually spatially distributed and composed of multiple individual foundation elements, e.g., footings or piles. The foundations can thus be designed with some level of redundancy in mind, to ensure that partial failure of the group does not result in the collapse of the entire system. Providing redundancy is costly and an economical approach to designing individual piles is of high interest in geotechnical and structural engineering. As pile foundations almost always appear in groups, with sometimes large numbers of piles having a strong potential for high levels of redundancy, the remainder of this paper will refer to pile foundation systems as the dominant example. However, it is emphasized that the results of this paper can be applied to any foundation system, composed of multiple individual foundations, so long as the covariance structure between foundations and between loads can be estimated or assumed.

This paper thus investigates the reliability of an example pile system, made up of $n_p$ piles, for various load and resistance statistics, and establishes a relationship between the reliability of this example foundation system and that of its individual components. For a target foundation system reliability, and considering correlations between individual pile loads and resistances, the required reliability of an individual pile can be determined using the theoretical and simulation results presented here. Individual piles can then be designed so that they collectively achieve the required pile system reliability. In other words, a key question answered here is: at what level of reliability, $\beta_i$, should individual piles be designed to successfully achieve a target system reliability $\beta_{sys}$?

Reliability of multi-component redundant systems has been studied by numerous researchers over the years. A common technique is load-sharing in which, as components fail one by one, the total load applied to the system is redistributed amongst the surviving components (see, e.g., Ang and Tang 1984, vol. II, example 7.10). The most common load-sharing approaches may be classified into equal load-sharing, tapered load-sharing, local load-sharing, nearest-neighbor load-sharing, and hybrid load-sharing (Durham et al. 1997). The widely used Daniels system (Daniels 1945) assumes equal load-sharing where all components share equal parts of the total load. That is, in a Daniels system comprising a set of $n_p$ components having independent and identically distributed resistances, $R_i$, $i = 1, \ldots, n_p$, subjected to random total load $F_T$, each component supports load $F_T/n_p$ and fails if $R_i < F_T/n_p$.

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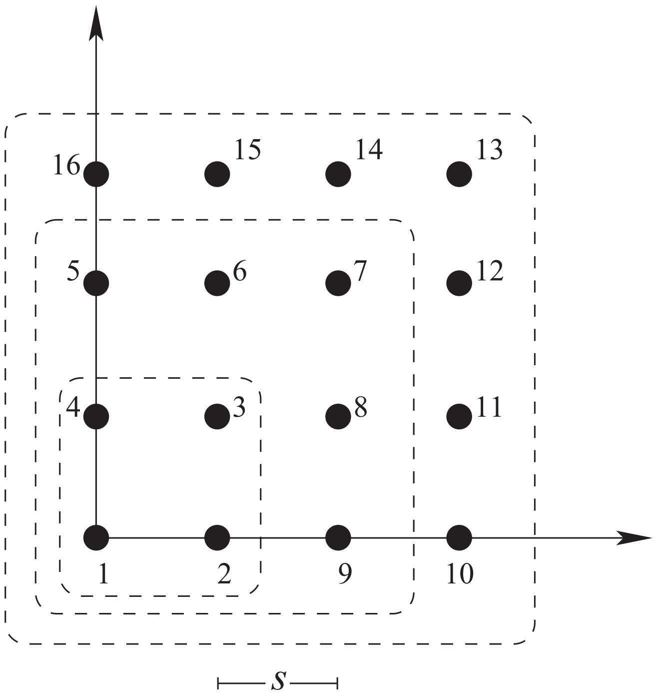
**Fig. 1.** Plan view of pile locations.




Generally speaking, the loads applied to each pile are often not equal, depending, as they do, on the pile load–settlement curve and the stiffness of the supported structure, as well as on how the random load is distributed within the supported structure. In addition, independent and identically distributed resistances will not realistically represent a multi-component pile system because pile resistances will often be strongly correlated. Strong correlation may arise, for example, if the piles are reasonably close together and thus founded in similar ground conditions. Recognizing these facts, the redundancy model studied here assumes random individual pile loads, $F_i$, $i = 1, \ldots, n_p$, and introduces correlation amongst the random resistances, $R_i$, and amongst the random loads, $F_i$. The correlation between the resistances is due to the persistence in ground properties over space, while the correlation between loads may arise if the supported structure is relatively stiff, so that loads are more equally distributed to the piles. Figure 1 illustrates the arrangement of the piles for cases where $n_p = 4$, 9, and 16, with $s$ denoting the pile spacing. Note that while the case where $n_p = 4$ is not usually considered to comprise a redundant pile system, it is included in this study as a lower bound on redundancy.

It is assumed that if the pile's resistance is less than the load imposed on it by the structure, i.e., $R_i < F_i$, then the pile "fails" in some sense. To examine the interaction between pile and system failure, a clear definition of what is meant by each is required. It is assumed that system failure occurs if the settlement of all supporting piles exceeds some critical amount. As this paper does not attempt to model the load–displacement nature of the ground, nor the stiffness characteristics of the supported structure (both of which would be very case-specific), system failure must instead be defined as when all $n_p$ piles are subjected to loads that exceed their resistances. The concept of settlement is not, however, entirely abandoned, because failure of individual piles will need to take the idea of settlement into account. For example, suppose that a realization of the random pile resistance is $R_i = 20$ kN, but that the realization of the random load applied to the pile is $F_i = 25$ kN. In this case, the pile is overloaded by 5 kN and will displace into the ground as a result. As the pile settles, an increasing portion of the original load will be redistributed to adjacent piles, the actual proportion being dependent on the stiffness of the supported structure and the nature of the load–displacement curve associated with the pile. In other words, in the event that $R_i < F_i$, the final load supported by the pile will be assumed to be $(1 - a)R_i$, where $a$ gives the fraction of the applied load that is "lost" once the pile's ultimate capacity has been exceeded; $a$ is 0.0 if the load–resistance curve for the pile–structure combination remains completely flat after the ultimate capacity, $R_i$, has been reached (i.e., the ultimate pile capacity, or more specifically, its sustained load, remains constant regardless of additional pile displacement into the ground) and is 1.0 if the load sustained by the pile reduces to zero after failure. It is also conceivable that $a$ could be negative, implying that the load sustained by the pile actually increases after its nominal failure. Here is where the interaction between settlement, load, and system failure again must be reconsidered. As mentioned above, it is assumed that an individual pile fails if it has $R_i < F_i$. The interpretation of this event in terms of the foundation–structural system is that if a pile has failed, then it has settled excessively. If all piles fail, then they all settle excessively, and the system fails. It is the process of settling that leads to sharing the pile's load to adjacent piles according to the factor $a$. It is further assumed that the load applied to a failed pile by the structure will never exceed the pile's original resistance (i.e., $a = 0$) and will often be less than the original resistance (i.e., $a > 0$). In other words, if $R_i < F_i$, then the excess structural load must be distributed to the other piles — the pile that failed cannot attract more load than $R_i$, due to the stiffness of the supported structure and the assumed excessive settlement of the pile, and will often end up attracting less load than its original resistance.

In general, lower values of $a$ lead to safer pile systems, because the pile continues to support at least some of the applied load after "settlement failure" occurs. In turn, this means that small values of $a$ give lower bounds on the failure probability of the foundation system. As discussed previously, only $0 \le a \le 1$ will be considered in this paper, with the residual "resistance" of a pile after failure assumed to be $(1 - a)R_i$.

If $m$ out of $n_p$ piles fail, then the remaining $n_p - m$ piles each support their initial applied load, $F_i$, as well as any excess load, $\Delta F$, due to failure of the other $m$ piles. That is

$$\Delta F = \frac{1}{n_p - m}\sum_{j=1}^{m}[F_j - (1 - a)R_j] \tag{1}$$

where it is assumed in eq. (1) that the piles have been numbered (or sorted) so that the first $j = 1, \ldots, m$ piles have failed (i.e., $R_j < F_j$ for $j = 1, \ldots, m$). The revised load on the $i$th ($i = m + 1, \ldots, n_p$) initially unfailed pile then becomes

$$F_i' = F_i + \Delta F \tag{2}$$

Note that it is assumed here that the load that is shed from failed piles is shared equally between all remaining piles in the foundation system. That is, the stiffness of the supported structure is assumed to be such that it leads to equal load-sharing of the loads not supported by all failed piles. Note also that the fraction of ultimate resistance lost by each pile after its failure (i.e., $R_i < F_i$), given by the parameter $a$, is assumed to be the same for all piles.

With the above in mind, this paper examines the reliability of a pile system for various levels of pile redundancy and load and resistance statistics using both theory and simulation. A relationship between the reliability of a pile system and the reliability of its individual components is established. The results can be used to safely, yet cost-effectively, design individual piles to achieve a target system reliability index, $\beta_{sys}$.

The paper is organized as follows. First, a random point process is presented for a system of $n_p$ spatially distributed piles, followed by a description of the corresponding simulation model. The reliability-based pile design approach is then described and a theoretical model is presented. The results are discussed and design implications illustrated by an example. The final section presents the main conclusions arising from the paper.

## Random point process

A random point process, $X(\underset{\sim}{x}_i)$, is a collection of random variables $X_1 = X(\underset{\sim}{x}_1)$, $X_2 = X(\underset{\sim}{x}_2)$, ..., whose values are associated with a discrete set of spatial locations $\underset{\sim}{x}_i$, $i = 1, \ldots, n_p$. The values in a random point process may be spatially correlated, and the spatial dependence is characterized by a correlation structure, which here is specified through a correlation function parameterized by a correlation length, $\theta$. In this paper, an isotropic exponentially decaying Markov correlation function is used, defined by

$$\rho(\tau_{ij}) = \exp\left\{\frac{-2|\tau_{ij}|}{\theta}\right\} \tag{3}$$

where $\tau_{ij}$ is the distance between any two points, $X_i$ and $X_j$, and $\theta$ is the correlation length (Fenton and Griffiths 2008). Small values of correlation length, (i.e., $\theta \to 0$), lead to uncorrelated points in the random point process where

$$\rho(\tau_{ij}) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{4}$$

Conversely, for completely correlated random variables, (i.e., $\theta \to \infty$), the correlation coefficient becomes

$$\rho(\tau_{ij}) = 1 \qquad \text{for all } i, j \tag{5}$$

A lognormal distribution is commonly used for modeling engineering properties due to its non-negative nature and its simple relationship with the normal distribution. In particular, a lognormal point process can be easily produced through a simple transformation of a Gaussian random point process. If $X$ is lognormal with mean and standard deviation $\mu_X$ and $\sigma_X$, then ln$X$ is normal with parameters

$$\begin{aligned} \sigma^2_{\ln X} &= \ln(1 + v^2_X) \\ \mu_{\ln X} &= \ln(\mu_X) - \frac{1}{2}\sigma^2_{\ln X} \end{aligned} \tag{6}$$

where $v_X = \sigma_X/\mu_X$ is the coefficient of variation of $X$. In this research, both load, $F$, and resistance, $R$, are assumed to be lognormally distributed random variables. This implies that ln$F$ and ln$R$ are both normally distributed with parameters given by eq. (6) (where the subscript $X$ is suitably replaced by either $R$ or $F$). Furthermore, both load and resistance are spatially varying random variables with an additional parameter being their correlation lengths, $\theta_{\ln F}$ and $\theta_{\ln R}$, respectively, replacing $\theta$ in eq. (3).

### Simulation model

Various random number generation algorithms exist of which the "covariance matrix decomposition" (CMD, see, e.g., Fenton and Griffiths 2008) method is employed in this research to provide realizations of the random load and resistance values. CMD is an exact method of producing realizations of a discrete random point process (i.e., at the pile locations) having prescribed mean, $\mu_{\ln X}$, and covariance matrix, $\underset{\approx}{C}$. The covariance matrix has elements $C_{ij} = \rho_{ij}\sigma_{\ln Xi}\sigma_{\ln Xj}$, $i, j = 1, 2, \ldots, n_p$, which give the covariance between any pair of points separated by lag distance $\tau_{ij}$, where $\rho_{ij} = \rho(\tau_{ij})$ (see eq. (3)). For a stationary random point process, having spatially constant mean and variance, the covariance matrix $\underset{\approx}{C}$ simplifies to having elements

$$C_{ij} = \begin{cases} \sigma^2_{\ln X} & \text{for } i = j \\ \sigma^2_{\ln X}\rho_{ij} & \text{for } i \neq j \end{cases} \tag{7}$$

Because $\underset{\approx}{C}$ is a positive definite covariance matrix, then a normally distributed (Gaussian) random point process, $\underset{\sim}{G}$, having elements $G_i = G(\underset{\sim}{x}_i)$, can be produced according to

$$\underset{\sim}{G} = \underset{\sim}{\mu}_{\ln X} + \underset{\approx}{L}\underset{\sim}{Z} \tag{8}$$

where $\underset{\sim}{x}_i$ is the spatial location of the $i$th point, $\underset{\approx}{L}$ is a lower triangular matrix satisfying $\underset{\approx}{L}\underset{\approx}{L}^T = \underset{\approx}{C}$ (obtained using Cholesky decomposition), and $\underset{\sim}{Z}$ is a vector of $n_p$ independent standard normal random variables (zero mean, unit variance). The lognormal random point process, $\underset{\sim}{X}$, is obtained from the normal process, $\underset{\sim}{G}$, using the following transformation:

$$\underset{\sim}{X} = \exp\{\underset{\sim}{G}\} \tag{9}$$

While CMD is simple and exact, it is inefficient for a large number of points. For example, a point process of size $n_p$ requires a covariance matrix of size $n_p^2$, which can become numerically challenging if $n_p$ is large (e.g., in excess of about 200). In this study, $n_p \leq 16$, which is easily managed by CMD.

## Design approach

As mentioned, loads and resistances are assumed to be lognormally distributed. An individual pile is initially subjected to individual random load $F_i$ having mean $\mu_{F_i}$ and coefficient of variation $v_{F_i}$. The first step is to determine the required mean pile resistance, $\mu_{R_i}$, using reliability-based design concepts. That is, the pile is to be designed to successfully support the initial individual load, $F_i$, with some target reliability index, $\beta_i$, i.e.,

$$\begin{aligned} P[R_i > F_i] &= P[R_i/F_i > 1] = P[\ln(R_i/F_i) > 0] = P[\ln W_i > 0] \\ &= \Phi\left(\frac{\mu_{\ln W}}{\sigma_{\ln W}}\right) = \Phi(\beta_i) \end{aligned} \tag{10}$$

where $\Phi$ is the standard normal cumulative distribution function, and $\beta_i = \mu_{\ln W}/\sigma_{\ln W}$ is the reliability index for an individual pile. In eq. (10), the quantity $W_i$ is defined as the ratio of pile resistance to applied load, and as such, is random and also follows a lognormal distribution so that

$$\ln W_i = \ln\left(\frac{R_i}{F_i}\right) = \ln R_i - \ln F_i \tag{11}$$

is normal with parameters

$$\begin{aligned} \mu_{\ln W} &= \mu_{\ln R_i} - \mu_{\ln F_i} \\ \sigma^2_{\ln W} &= \sigma^2_{\ln R_i} + \sigma^2_{\ln F_i} \end{aligned} \tag{12}$$

where independence between the random variables $R_i$ and $F_i$ (or ln$R_i$ and ln$F_i$) was assumed to compute $\sigma^2_{\ln W}$. With reference to eq. (6), the mean and variance of resistance, $R_i$, are

$$\begin{aligned}\sigma^2_{\ln R_i} &= \ln\left(1+\nu^2_{R_i}\right)\\ \mu_{\ln R_i} &= \ln(\mu_{R_i}) - \frac{1}{2}\sigma^2_{\ln R_i}\end{aligned} \tag{13}$$

Similarly,

$$\begin{aligned}\sigma^2_{\ln F_i} &= \ln\left(1+\nu^2_{F_i}\right)\\ \mu_{\ln F_i} &= \ln(\mu_{F_i}) - \frac{1}{2}\sigma^2_{\ln F_i}\end{aligned} \tag{14}$$

Substituting eqs. (13) and (14) into eq. (12) gives

$$\begin{aligned}\sigma^2_{\ln W} &= \ln\left[\left(1+\nu^2_{R_i}\right)\left(1+\nu^2_{F_i}\right)\right]\\ \mu_{\ln W} &= \ln\left[\frac{\mu_{R_i}/\mu_{F_i}}{\sqrt{\left(1+\nu^2_{R_i}\right)/\left(1+\nu^2_{F_i}\right)}}\right]\end{aligned} \tag{15}$$

The individual reliability index is obtained using eq. (15) as follows:

$$\beta_i = \frac{\mu_{\ln W}}{\sigma_{\ln W}} = \frac{\ln\left[\dfrac{\mu_{R_i}/\mu_{F_i}}{\sqrt{\left(1+\nu^2_{R_i}\right)/\left(1+\nu^2_{F_i}\right)}}\right]}{\sqrt{\ln\left[\left(1+\nu^2_{R_i}\right)\left(1+\nu^2_{F_i}\right)\right]}} \tag{16}$$

Solving eq. (16) for $\mu_{R_i}$ gives

$$\begin{aligned}\mu_{R_i} &= \mu_{F_i}\exp\left\{\beta_i\sqrt{\ln\left[\left(1+\nu^2_{R_i}\right)\left(1+\nu^2_{F_i}\right)\right]}\right\}\sqrt{\left(1+\nu^2_{R_i}\right)/\left(1+\nu^2_{F_i}\right)}\\ &= \mu_{F_i}\exp(\beta_i\sigma_{\ln W})\sqrt{\left(1+\nu^2_{R_i}\right)/\left(1+\nu^2_{F_i}\right)}\end{aligned} \tag{17}$$

which indicates that the design mean resistance, $\mu_{R_i}$, depends on mean individual load, $\mu_{F_i}$, individual target reliability index, $\beta_i$, as well as load and resistance coefficients of variation, $\nu_{F_i}$ and $\nu_{R_i}$, respectively.

Once a design mean resistance, $\mu_{R_i}$, is obtained via eq. (17), the simulation process is carried out as follows:

1. Two lognormal random point processes, each of size $n_p$, are generated for the loads, $F_i$, and resistances, $R_i$, associated with individual piles in a pile system arranged as depicted in Fig. 1. The resistance and load distribution parameters are given by eqs. (13) (with eq. (17)) and (14), respectively.
2. Individual piles are ranked from those with the smallest $R_i/F_i$ ratio to those with the largest ratio.
3. The system fails if all $R_i/F_i$ ratios are less than 1. Otherwise, if the first $m$ piles (after ranking above) have $R_i/F_i$ ratios less than 1, it means that these piles have been overloaded and cannot support their full applied load $F_i$ (assuming $a \geq 0$; see earlier discussion). In this case, the residual load that is not carried by these $m$ piles must be distributed to the remaining $n_p - m$ piles according to eqs. (1) and (2).
4. Repeat steps 2 and 3 after redistribution of loads from failed piles, updating the number of failed piles, $m$, until all failed piles have been found.
5. The system survives if any $R_i/F_i$ ratios exceed 1, otherwise the system fails.

The above process is repeated $n_{sim}$ times after which the system failure probability is estimated using

$$p_f \approx n_f/n_{sim} \tag{18}$$

where $n_f$ is the number of realizations resulting in a system failure and $n_{sim}$ is the total number of realizations.

## Theoretical model

A theoretical approach to assessing the reliability of a pile system is relatively easily derived when $a = 0$. In this case,

$$\begin{aligned}p_f &= P\left[\sum_{i=1}^{n_p} R_i < \sum_{i=1}^{n_p} F_i\right] = P\left[\sum_{i=1}^{n_p} R_i - \sum_{i=1}^{n_p} F_i < 0\right]\\ &= P[Y<0] = \Phi\left(\frac{-\mu_Y}{\sigma_Y}\right) = \Phi(-\beta_{sys})\end{aligned} \tag{19}$$

where $\beta_{sys} = \mu_Y/\sigma_Y$ is the reliability index for the pile system. The quantity $Y$ is defined as the difference between the total load applied to the pile system and the total resistance provided by the system.

As an aside, when $a \neq 0$ the analytical solution involves including interdependent Bernoulli random variables, $X_i$, to monitor which piles fail in the system, according to

$$p_f = P\left[\sum_{i=1}^{n_p} R_i X_i < \sum_{i=1}^{n_p} F_i\right] = P\left[\sum_{i=1}^{n_p} R_i X_i - \sum_{i=1}^{n_p} F_i < 0\right] \tag{20}$$

where

$$X_i = \begin{cases}1-a & \text{if } R_i < F_i \text{(pile failure)}\\ 1 & \text{otherwise}\end{cases} \tag{21}$$

As far as the authors are aware, the only practical way to solve eq. (20) is through simulation.

Restricting attention to the $a = 0$ case, eq. (19) can be solved explicitly by noting that the random variable

$$Y = \sum_{i=1}^{n_p} R_i - \sum_{i=1}^{n_p} F_i \tag{22}$$

is at least approximately normally distributed according to the central limit theorem because it involves a sum of up to $2n_p$ independent random variables. If strong correlations between pile resistances and between applied loads exist, then $Y$ may be a sum of as little as two independent random variables, $R_i$ and $F_i$, for any $i$, in which case the central limit theorem normal approximation deteriorates. However, for lognormally distributed loads and resistances whose coefficients of variation are less than about 0.3 (as assumed here), the normal approximation to even $Y = n_p(R_i - F_i)$ will be quite accurate. Using these arguments, $Y$ is assumed to be normally distributed with parameters

$$\begin{aligned}\mu_Y &= E\left[\sum_{i=1}^{n_p} R_i - \sum_{i=1}^{n_p} F_i\right] = n_p(\mu_{R_i} - \mu_{F_i})\\ \sigma_Y^2 &= \sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\text{Cov}[R_i,R_j] + \sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\text{Cov}[F_i,F_j]\\ &= \sigma^2_{R_i}\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\rho_{R_{ij}} + \sigma^2_{F_i}\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\rho_{F_{ij}} = n_p^2\left(\sigma^2_{R_i}\gamma_R + \sigma^2_{F_i}\gamma_F\right)\end{aligned} \tag{23}$$

where stationarity and independence between the random variables $R_i$ and $F_i$ were used to compute $\sigma_Y^2$. The quantity $\gamma_R$ in eq. (23) is a variance reduction factor defined as

(24) $$\gamma_R = \frac{1}{n_p^2}\sum_{i=1}^{n_p}\sum_{j=1}^{n_p}\rho_{R_{ij}}$$

where $\rho_{R_{ij}} = \rho_R(\tau_{ij})$ is the correlation coefficient acting between the $i$th and $j$th pile resistances, $R_i$ and $R_j$ (see eq. (3) for how this correlation coefficient is defined). Equation (24) becomes $\gamma_R = 1/n_p$ for uncorrelated pile resistances, and $\gamma_R = 1$ for completely correlated pile resistances. Thus, the variance reduction factor takes values in the range $1/n_p \le \gamma_R \le 1$ depending on the level of dependency amongst the piles in a pile system. An entirely similar equation (and discussion) exists for the variance reduction factor $\gamma_F$ simply by replacing the subscript $R$ by $F$.

Note that using eqs. (17) and (23), eq. (19) reduces to

(25) $$p_f = \Phi\left(\frac{1-c}{\sqrt{v_{R_i}^2 c^2 \gamma_R + v_{F_i}^2\gamma_F}}\right)$$

where

(26) $$c = \exp\left\{\beta_i\sqrt{\ln\left[\left(1+v_{R_i}^2\right)\left(1+v_{F_i}^2\right)\right]}\right\}\sqrt{\left(1+v_{R_i}^2\right)/\left(1+v_{F_i}^2\right)}$$

is a function of just $\beta_i$, $v_{R_i}$, $v_{F_i}$ and independent of the means in both the load and resistance. This means that the choices of $\mu_{F_i}$ and $\mu_{R_i}$ make no difference to the results of this paper.

## Results and discussion

The objective of this section is to investigate how the individual reliability, $\beta_i$, relates to system reliability, $\beta_{sys}$. The parameters used in the more general Monte Carlo simulation study are listed in Table 1. The variables $\theta_F$ and $\theta_R$ are the correlation lengths used in eq. (3) to specify the correlation coefficients between the $\ln F_i$ values and between the $\ln R_i$ values, respectively, and an "intermediate" value is selected corresponding to moderate levels of correlation between piles. The cases where the correlation lengths become zero or infinite are considered shortly. The coefficient of variation of resistance, $v_{R_i} = 0.15$, is as used by Fenton et al. (2016). The coefficient of variation of the total load, $v_F = 0.1$, is derived by assuming that live and dead load coefficients of variation are $v_L = 0.27$ and $v_D = 0.1$ (Allen 1975), respectively, and that the dead to live load ratio is $R_{D/L} = 3.0$. Assuming that live and dead loads are independent leads to $v_{F_i} = \sqrt{v_L^2+(R_{D/L}v_D)^2}/(1+R_{D/L}) = \sqrt{0.27^2+9(0.1)^2}/4 = 0.1$.

The number of realizations, $n_{sim}$, required to determine the probability of system failure, $p_f = \Phi(-\beta_{sys})$, to within a relative error of 20% with 95% confidence is

(27) $$n_{sim} = \frac{1.96^2 p_f}{(0.2p_f)^2} = \frac{1.96^2}{0.04\Phi(-\beta_{sys})} \approx \frac{96}{\Phi(-\beta_{sys})}$$

For example, if $\beta_{sys} = 3$, then approximately 70 000 realizations are needed to estimate $p_f$ reasonably accurately. For any target $\beta_{sys}$, the task now is to determine the required corresponding $\beta_i$. This is accomplished through the following steps:

1. Initially guess that $\beta_i = 0$ (or some small value).
2. Compute the required design value of $\mu_{R_i}$ using eq. (17).
3. Estimate the probability of foundation system failure, $p_f$, according to the algorithm given in the previous section and eq. (18).
4. Compute $\beta'_{sys} = \Phi^{-1}(1-p_f)$.
5. If $\beta'_{sys} < \beta_{sys}$, then increase $\beta_i$ and repeat steps 2–5 until $\beta'_{sys} = \beta_{sys}$. Similarly, if $\beta'_{sys} > \beta_{sys}$, then $\beta_i$ must be decreased.

**Table 1.** Input parameters used in simulation.

| Parameter | Values considered |
|---|---|
| $n_p$ | 4, 9, 16 |
| $v_{F_i}$ | 0.1 |
| $v_{R_i}$ | 0.15 |
| $\theta_F/s = \theta_R/s$ | 1 |
| $a$ | 0, 0.5, 1.0 |

**Fig. 2.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, $a = 0$, and various number of piles, $n_p$, according to simulation results.

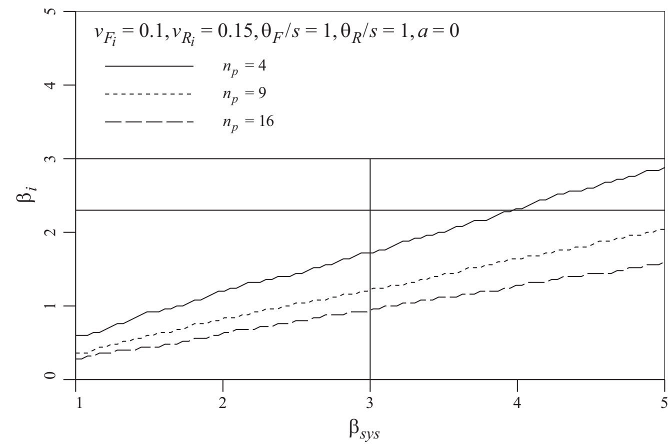
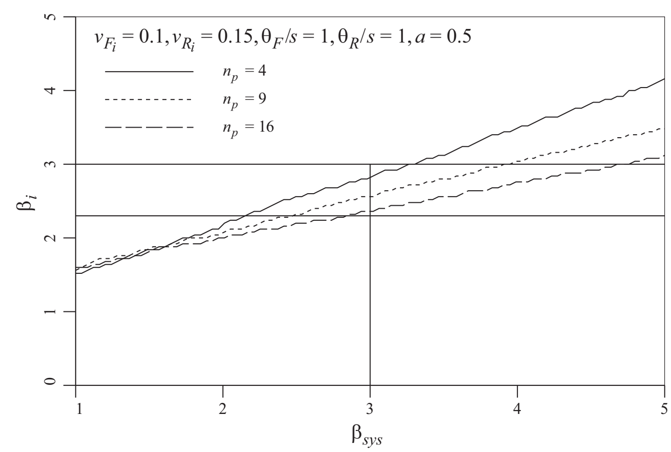


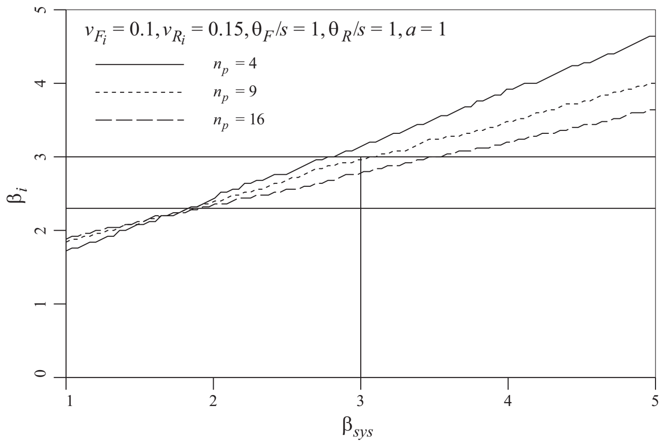

**Fig. 3.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, $a = 0.5$, and various number of piles, $n_p$, according to simulation results.




Figures 2–4 illustrate how the reliabilities of individual piles, $\beta_i$, relate to the system reliability, $\beta_{sys}$, for various values of $a$, and a moderate correlation length value of $\theta_F = \theta_R = s$, where $s$ is the center to center pile spacing. These figures can be used for design by drawing a vertical line at the target system reliability index, $\beta_{sys}$, and then reading off the required $\beta_i$ value for a given $n_p$. For example, for a foundation system consisting of $n_p = 9$ piles, and a moderate target system reliability of $\beta_{sys} = 3.0$, corresponding to $p_f \approx 1/1000$, the recommended single pile reliability index is given by Fig. 2 to be $\beta_i = 1.22$ for $a = 0$. When $a = 0$ in eq. (1), the pile resistance is assumed to never be less than $R_i$, even if the pile "fails" ($R_i < F_i$, see discussion above).

If the pile resistance is assumed to go to zero as soon as it fails, i.e., $a = 1$, then Fig. 4 recommends that $\beta_i = 2.96$ should be used in the design of an individual pile when $n_p = 9$. In other words, if the pile strength disappears completely upon failure, so that the entire load originally supported is shed to adjacent piles, then the

**Fig. 4.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, $a = 1$, and various number of piles, $n_p$, according to simulation results.




pile system approaches a “weakest link” problem, more so as $n_p$ decreases. The “weakest link” problem involves a system that fails when its weakest link fails. In the case when $a = 1$, the shedding of a large amount of load to adjacent piles as soon as $R_i < F_i$, for a single pile, often leads to failure of all of the piles, so that system failure is often governed by the weakest pile. Note, however, that this is not quite a “weakest link” problem, as there is always the possibility that even a single remaining pile could support the entire applied load.

Alternatively, the theoretical model described in the “Theoretical model” section could be used to estimate the required $\beta_i$ for a target $\beta_{sys}$ when $a = 0$ in eq. (1). Figures 5–7 illustrate such an analysis, generated in the same fashion as in the simulation using the five-step algorithm described earlier in this section, except that in step 3, the foundation system failure probability is estimated directly via eq. (25). The individual pile reliability estimated by theory is superimposed on the simulation-based plots, allowing a direct comparison of the two methods at low ($\theta/s \to 0$, $\gamma_R = \gamma_F = 1/n_p$, in eq. (23)), moderate ($\theta/s = 1$), and high ($\theta/s \to \infty$, $\gamma_R = \gamma_F = 1$ in eq. (23)) levels of dependency amongst piles in the pile system. The excellent agreement between the two methods illustrated in Figs. 5–7, where the theoretical predictions lie directly on top of the simulation results, strongly suggests that the theoretical model described in the “Theoretical model” section is a legitimate replacement for simulation when $a = 0$ in eq. (1). The theoretical model is simple, easy to use, and eliminates the need for simulation. Note that, for instance, at a relatively high target system reliability of $\beta_{sys} = 5$, $n_{sim} \approx 334\,901\,355$ realizations are needed to estimate $\beta'_{sys}$ to within a relative error of 20% at 95% confidence (see eq. (27) and the algorithm to find $\beta_i$ above). For such a case, the Monte Carlo simulation is very time-consuming.

It is evident from Fig. 7 that, for a completely correlated pile system, the individual pile failure and the pile system failure become equal and thus, the individual piles must be designed at target system reliability $\beta_i = \beta_{sys}$ to maintain the target system reliability, $\beta_{sys}$. In other words, the worst-case correlation length for this analysis happens to be $\theta/s \to \infty$ ($\gamma_R = \gamma_F = 1$ in eq. (23)) when the piles in a pile system are fully correlated. When the correlation length is infinity, the foundation resistances are all equal, as are the loads (if $\theta_F = \theta_R$, as assumed here). This means that if one foundation fails, they will all fail. In other words, it is equivalent to having just a single foundation. This is not generally very realistic, although might occur on very uniform clay deposits, for example. It is also unlikely that the loads applied to the piles will also all be the same.

Generally speaking, a reliability index of $\beta_i = 3.0$ ($p_f \approx 1/1000$) is prescribed in geotechnical design practice as the target reliability

**Fig. 5.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, and various number of piles, $n_p$, according to theoretical and simulation results (corresponding to $a = 0$) for small $\theta/s$.

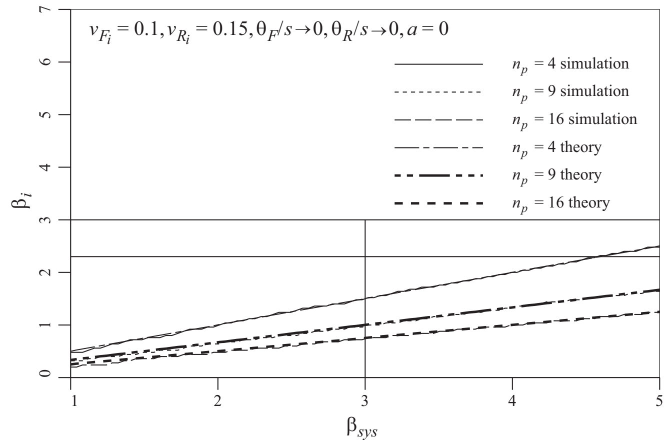



**Fig. 6.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, and various number of piles, $n_p$, according to theoretical and simulation results (corresponding to $a = 0$).

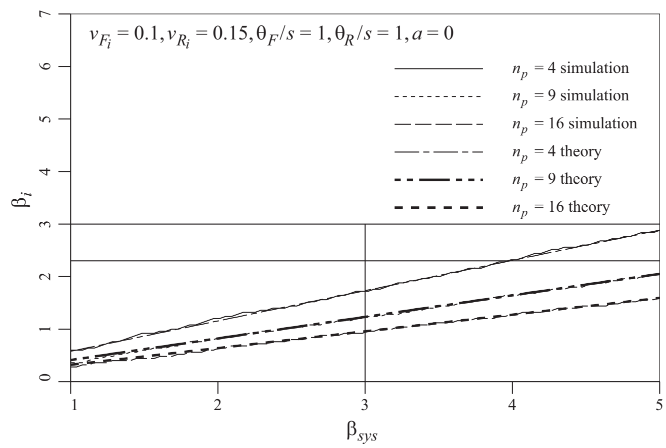
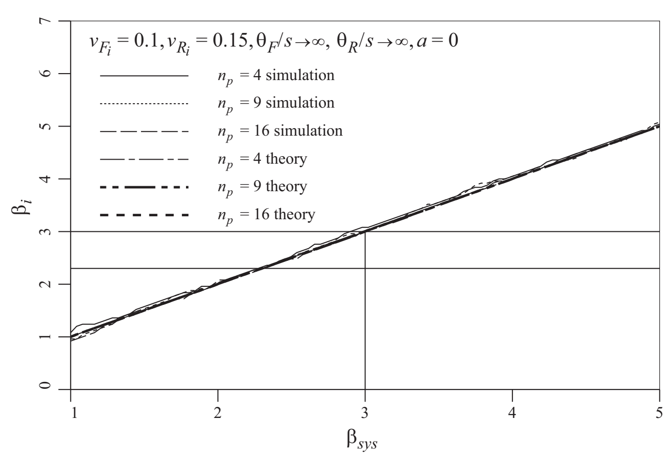



**Fig. 7.** Plot of $\beta_i$ versus $\beta_{sys}$ for $v_{F_i} = 0.1$, $v_{R_i} = 0.15$, and various number of piles, $n_p$, according to theoretical and simulation results (corresponding to $a = 0$) for large $\theta/s$.




index of an individual pile in nonredundant pile systems ($n_p \le 4$), and $\beta_i = 2.3$($p_f \approx 1/100$) for redundant pile systems ($n_p \ge 5$) (Zhang et al. 2001; NCHRP 2004; Allen 2005; NCHRP 1991). According to Zhang et al. (2001), a $\beta_{sys}$ value of 3.0 requires a $\beta_i$ = 2.0–2.8 for redundant systems, which is in agreement with Fig. 3 results generated for $a = 0.5$. In other words, taking into account about half of the resistance of the failed piles will require an individual reliability index of $\beta_i$ = 2.3–2.5 to achieve a target system reliability of $\beta_{sys} = 3.0$ for redundant systems ($n_p \geq 5$).

**Example**

Consider a foundation system consisting of $n_p = 9$ piles, and a moderate target system reliability of $\beta_{sys} = 3.0$, corresponding to $p_f \approx 1/1000$. Assume that the coefficients of variation and correlation lengths at the site are as specified in Table 1 and that piles continue to support load $R_i$ even after they have failed ($a = 0$). In this case, the recommended single pile reliability index is given by Fig. 2 to be $\beta_i = 1.22$. If the mean total load on the foundation system is 1000 kN, then the mean load applied to each pile can be assumed to be $\mu_{F_i} = 1000/9$ so that the mean pile resistance required to achieve $\beta_i = 1.22$ is given by eq. (17) to be

$$\mu_{R_i} = \left(\frac{1000}{9}\right)\exp\left\{1.22\sqrt{\ln[(1+0.15^2)(1+0.1^2)]}\right\} \times \sqrt{\frac{(1+0.15^2)}{(1+0.1^2)}} = 139.2\ \text{kN} \tag{28}$$

Supposing that the piles are founded in a frictional soil, the characteristic pile resistance, $\hat{R}_u$, is given by Fenton and Naghibi (2011) to be

$$\hat{R}_u = \frac{1}{2}pc\gamma H^2(1-\sin\hat{\phi})\tan b\hat{\phi} \tag{29}$$

where $p$ is the effective pile perimeter length, $c$ is a factor relating to the earth pressure coefficient, $\gamma$ is the soil unit weight, $\hat{\phi}$ is the characteristic soil friction angle, and $b$ is the pile interface friction angle coefficient. For this example, assume that $p = 1.0$ m, $c = 1.2$ for low-displacement piles (see Das 2000), $\gamma = 18$ kN/m$^3$, and $\hat{\phi} = 30°$.

The characteristic pile resistance, predicted for example by eq. (29), is often an imperfect estimate of the mean pile resistance and the bias factor, defined as $k_R = \mu_{R_i}/\hat{R}_u$, reflects the mean "bias" between the mathematical model and the actual mean pile resistance. Using this bias factor, eq. (29) can be inverted to solve for the pile length required to achieve an individual pile reliability of $\beta_i = 1.22$

$$H = \sqrt{\frac{2\mu_{R_i}/k_R}{pc\gamma(1-\sin\hat{\phi})\tan b\hat{\phi}}} \tag{30}$$

If, as assumed by Fenton and Naghibi (2011), eq. (29) is an unbiased estimate of the mean pile resistance, then $k_R = 1$, and the design pile length for this example becomes

$$H = \sqrt{\frac{2(139.2)/1.0}{(1.0)(1.2)(18)(1-\sin 30)\tan(0.7\times 30)}} = 8.2\ \text{m} \tag{31}$$

## Conclusions

In geotechnical design, foundations are usually designed individually. Group effects are usually only considered with respect to overall block behavior, where the pile group is viewed as a single equivalent pile, or load-carrying group efficiency (see, e.g., the Canadian Geotechnical Society 2006). If piles are viewed as components in a system and the system fails only if all of the components fail, then it is natural to ask what reliability the components must have to achieve a certain target reliability in the system. This paper specifically considers this question. The piles (or foundation elements; the results could also be applied to other types of foundations) are deemed to be mutually correlated components resisting a set of random (also correlated) loads. The foundation system fails if the loads exceed the foundation resistances at all foundations.

A difficulty with the model is how failure is actually defined. Despite the fact that the load–displacement characteristics of the foundation–structural system are not explicitly modeled in this paper, the concept of failure being defined by excessive displacement must be considered. This is because many foundations will continue to show increasing resistance as displacement increases, so that system failure, based strictly on foundation resistance without regarding displacement, becomes extremely unlikely. However, the system may have long since failed due to excessive displacement of the supporting foundation.

In this paper, a foundation is deemed to have failed if the load applied to it exceeds its capacity. In the event that this takes place, the foundation is assumed to have displaced sufficiently to cause structural failure if all other piles reach the same state. The load applied to a failed pile is assumed to be some fraction of the pile's original ultimate capacity, $(1-a)R_i$, ranging from the ultimate capacity when $a = 0$ to zero when $a = 1$, the actual value depending on the amount of load transferred to other piles through the structure's stiffness characteristics after displacement of the pile.

The value of $a$ has a significant effect on the individual foundation reliability, $\beta_i$, required to achieve a target system reliability, $\beta_{sys}$. The higher the value of $a$, the lower the residual pile capacity, so the higher the required individual reliability index. When $a = 1$, the pile has lost all capacity upon failure and the individual reliability index approaches the system reliability index. Apparently, assuming $a = 1$ would be quite a conservative assumption. At the other extreme, when it is assumed that $a = 0$ (or less) so that the load attracted by the pile remains at its original ultimate capacity (or higher), the required individual pile reliability index becomes quite low. See, for example, Fig. 2 where a nine-pile system has $\beta_i \approx 1.2$ for $\beta_{sys} = 3.0$.

The actual value of $a$ that should be used depends entirely on the load–displacement characteristics of the foundation–structural system, as well as on what constitutes failure in the structure. Both of these issues are very site-specific problems. Common practice suggests that a value of $a = 0.5$ is reasonably conservative and corresponds to $\beta_i$ values that are generally recommended in the literature. If this is so, the results suggested by Fig. 3 may well be appropriate for use in code development.

One further conclusion of this study is that the determination of the relationship between geotechnical redundancy and system reliability is quite site-specific, depending very much on the definition of structural failure and the structure–foundation load–displacement relationships. The results presented by this paper nevertheless provide insight into the geotechnical reliability required for redundant foundation systems.

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## References

Allen, D.E. 1975. Limit states design – a probabilistic study. Canadian Journal of Civil Engineering, **2**(1): 36–40. doi:10.1139/l75-004.

Allen, T.M. 2005. Development of geotechnical resistance factors and downdrag load factors for LRFD foundation strength limit state design. U.S. Department of Transportation, Federal Highway Administration, FHWANHI05052, Washington, D.C.

Ang, H.-S., and Tang, W.H. 1984. Probability concepts in engineering planning and design. Vol. II. John Wiley & Sons, New York.

Canadian Geotechnical Society. 2006. Canadian foundation engineering manual. 4th ed. Canadian Geotechnical Society, Montréal, Que.

Daniels, H.E. 1945. The statistical theory of the strength of bundles of threads, Part I. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, **183**(995): 405–435. doi:10.1098/rspa.1945.0011.

Das, B.M. 2000. Fundamentals of geotechnical engineering. Brooks/Cole, Pacific Grove, Calif.

Durham, S.D., Lynch, J.D., Padgett, W.J., Horan, T.J., Owen, W.J., and Surles, J. 1997. Localized load-sharing rules and Markov–Weibull fibers: a comparison of microcomposite failure data with Monte Carlo simulations. Journal of Composite Materials, **31**: 1856–1882. doi:10.1177/002199839703101805.

Fenton, G.A., and Griffiths, D.V. 2008. Risk assessment in geotechnical engineering. John Wiley & Sons, New York.

Fenton, G.A., and Naghibi, M. 2011. Geotechnical resistance factors for ultimate limit state design of deep foundations in frictional soils. Canadian Geotechnical Journal, **48**(11): 1742–1756. doi:10.1139/t11-068.

Fenton, G.A., Naghibi, F., Dundas, D., Bathurst, R.J., and Griffiths, D.V. 2016. Reliability-based geotechnical design in 2014 Canadian Highway Bridge Design Code. Canadian Geotechnical Journal, **53**(2): 236–251. doi:10.1139/cgj-2015-0158.

NCHRP. 1991. Manuals for the design of bridge foundations. National Cooperative Highway Research Program. Transportation Research Board, NRC, Washington, D.C. Report 343.

NCHRP. 2004. Load and resistance factor design (LRFD) for deep foundations. National Cooperative Highway Research Program, Transportation Research Board, NRC, Washington, D.C. Report 507.

Zhang, L., Tang, W.H., and Ng, C.W.W. 2001. Reliability of axially loaded driven pile groups. Journal of Geotechnical and Geoenvironmental Engineering, **127**(12): 1051–1060. doi:10.1061/(ASCE)1090-0241(2001)127:12(1051).

## List of symbols

$a$ load-sharing factor
$b$ pile interface friction angle coefficient
$c$ earth pressure coefficient and variable in eq. (25)
$\mathrm{Cov}[X, Y]$ covariance between random variables $X$ and $Y$
$\underset{\approx}{\mathbf{C}}$ random point process covariance matrix with elements $C_{ij} = \rho_{ij}\sigma_{\ln Xi}\sigma_{\ln Xj}$
$\mathrm{E}[.]$ expectation operator
$F$ load
$F_i$ individual pile load (random)
$F_i'$ revised individual pile load ($= F_i + \Delta F$)
$F_\mathrm{T}$ true total load (random)
$\underset{\sim}{\mathbf{G}}$ Gaussian random point process $\left(= \underset{\sim}{\boldsymbol{\mu}}_{\ln X} + \underset{\approx}{\mathbf{L}}\underset{\sim}{\mathbf{Z}}\right)$
$G_i$ element of Gaussian random point process at the spatial location $\underset{\sim}{\mathbf{x}}_i$ $(= G(\underset{\sim}{\mathbf{x}}_i))$
$H$ design pile length
$k_R$ resistance bias factor $\left(= \mu_{R_i}/\hat{R}_\mathrm{u}\right)$
$\underset{\approx}{\mathbf{L}}$ lower triangular matrix used in CMD
$m$ number of failed piles out of $n_\mathrm{p}$ piles
$n_\mathrm{f}$ number of realizations resulting in a system failure
$n_\mathrm{p}$ number of piles in a pile system
$n_\mathrm{sim}$ number of simulations
$\mathrm{P}[.]$ probability operator
$p$ effective pile perimeter length
$p_\mathrm{f}$ probability of system failure
$R$ resistance
$R_\mathrm{D/L}$ dead to live load ratio
$R_i$ individual pile resistance (random)
$\hat{R}_\mathrm{u}$ characteristic (design) pile resistance
$s$ center-to-center pile spacing
$\nu_\mathrm{D}$, $\nu_\mathrm{L}$ dead and live load coefficient of variation, respectively
$v_{F_i}$ load coefficient of variation $\left(= \sigma_{F_i}/\mu_{F_i}\right)$
$v_{R_i}$ resistance coefficient of variation $\left(= \sigma_{R_i}/\mu_{R_i}\right)$
$v_X$ coefficient of variation of $X$
$W_i$ pile resistance to load ratio ($= R_i/F_i$)
$X_i$ element of a random point process or Bernoulli random variable
$\underset{\sim}{\mathbf{x}}$ spatial coordinate in two dimensions
$\underset{\sim}{\mathbf{X}}$ random point process
$Y$ normal random variable $\left(= \sum_{i=1}^{n_\mathrm{p}} R_i - \sum_{i=1}^{n_\mathrm{p}} F_i\right)$
$\underset{\sim}{\mathbf{Z}}$ standard normal random point process
$\beta_i$ reliability index of an individual pile
$\beta_\mathrm{sys}$ reliability index of the pile system $\beta'_\mathrm{sys} < \beta_\mathrm{sys}$
$\beta'_\mathrm{sys}$ trial reliability index of pile system
$\gamma$ effective unit weight of soil
$\gamma_F$ load variance reduction factor
$\gamma_R$ resistance variance reduction factor
$\theta$ correlation length of a random point process
$\theta_F$ correlation length of load random point process
$\theta_R$ correlation length of resistance random point process
$\Delta F$ excess load due to failure of $m$ piles
$\mu_{F_i}$ mean individual load
$\mu_{\ln F_i}$ mean of $\ln F_i$
$\mu_{\ln R_i}$ mean of $\ln R_i$
$\mu_{\ln W}$ mean of $\ln W$
$\mu_{\ln X_i}$ mean of $\ln X_i$
$\mu_{R_i}$ mean individual resistance
$\mu_X$ mean of $X$
$\rho$ correlation coefficient between two random variables
$\rho_{F_{ij}}$ correlation coefficient between loads $F_i$ and $F_j$
$\rho_R$ correlation coefficient between resistances
$\rho_{R_{ij}}$ correlation coefficient between resistances $R_i$ and $R_j$
$\sigma_{F_i}$ pile load standard deviation
$\sigma_{\ln F}$ standard deviation of $\ln F$
$\sigma_{\ln F_i}$ standard deviation of $\ln F_i$
$\sigma_{\ln R}$ standard deviation of $\ln R$
$\sigma_{\ln R_i}$ standard deviation of $\ln R_i$
$\sigma_{\ln W}$ standard deviation of $\ln W$
$\sigma_{\ln X}$ standard deviation of $\ln X$
$\sigma_{R_i}$ resistance standard deviation
$\sigma_X$ standard deviation of $X$
$\sigma_Y$ standard deviation of $Y$
$\tau_{ij}$ lag distance
$\hat{\phi}$ characteristic value of friction angle
$\Phi$ standard normal cumulative distribution function

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