

# Numerical analysis of the influence of porosity and void size on soil stiffness using random fields

D.V. Griffiths, Jumpol Paiboon, Jinsong Huang  
*Colorado School of Mines, Golden, CO, USA*

Gordon A. Fenton  
*Dalhousie University, Halifax, NS, Canada*

**ABSTRACT:** The purpose of the study is to investigate the influence of porosity and void size on equivalent geotechnical engineering properties. The paper uses random field theory to generate models of geomaterials containing spatially random voids with controlled porosity and void size. Analyses then use finite elements and Monte-Carlo simulations, where each simulation leads to an effective value of the property under investigation. The effective value is defined as the property that would have led to the same response if the geomaterial had been homogeneous. Although the methodology can be applied to a variety of soil properties, the paper concentrates on elastic stiffness. A novel finite element model is developed involving “tied freedoms” that allows analysis of an ideal element test and hence direct evaluation of the effective Young’s modulus and Poisson’s ratio. A foundation problem of a footing on an elastic foundation containing voids is also described.

## 1 INTRODUCTION

The motivation for this work came from a study of foundations resting on subsurface materials containing voids of variable porosity and size (Griffiths et al. 2011). The geology of the region consisted of karst which is a special type of landscape and subsurface characterized by the dissolution of soluble rocks, including limestone and dolomite. Given limited site investigation data and the spatial unpredictability of the underground void locations, a numerical study was conducted to determine the effective elastic properties of materials with randomly distributed voids.

The goal of the study was to generate charts giving guidance on the statistical distribution (mean, standard deviation and PDF) of effective Young’s modulus and Poisson’s ratio as a function of porosity and void size. The effective value is defined as the property that would have led to the same response if the geomaterial had been homogeneous.

The void size component of the study was intended to investigate whether two deposits with the same porosity but quite different void size distributions would perform differently.

Having established the statistical distributions of effective properties as mentioned above, probabilistic statements relating to foundation performance under design loads can then be made, e.g. the probability of excessive settlement for a foundation resting on subsurface materials with given porosity and void size.

## 2 PREVIOUS WORK

The macro-scale response of a homogeneous material which has a heterogeneous micro-structure has been studied by a number of investigators. The goal is to obtain the effective or equivalent properties at the macro-scale. An important objective of micro-mechanics is to link mechanical relations at different length scales ranging from finer to coarser length scales.

Under the assumption that the material behavior at the micro-scale is known and is representative of the entire heterogeneous material, a representative volume element (RVE) can be employed in the homogenization process. The concept of RVE was first introduced by Hill (1963) since when there have been many numerical simulations developed and applied to determine RVE size (e.g. Kulatilake 1985, Ning et al. 2008 and Esmaili et al. 2010).

Several theoretical models have been proposed for dealing with scale effects ranging from micro to macro levels. The Differential Method (Roscoe 1952) has a long and interesting history, and has been one of the most effective and widely used methods. The Composite Spheres Model (Hashin 1962) considered only a single inclusion and led to simple closed-form expressions. The Self Consistent Method (Budiansky 1965, Hill 1965) and the Generalized Self Consistent Method, formalized by Christensen and Lo (1979) involved embedding an inclusion phase directly into an infinite medium. Christensen and Lo (1979) explained that the final

form of this method can solve the spherical inclusion problem. Finally, the Mori-Tanaka (M-T) method as described by Benveniste (1987) has attracted a lot of interest but is quite different to the other methods. The M-T method involves complex manipulations of the field variables along with special concepts of strain and stress. Although there are many analytical models for estimating the effective elastic properties of a material containing voids, they are often limited to voids with simple shapes.

Numerical methods such as the Finite Element Method (FEM) or the Boundary Element Method (BEM) have been used to validate some of the theoretical approaches. Two major variables can be investigated in a realistic representation of a defective material; namely the volume and size of the voids or inclusions. Isida and Igawa (1991) considered several kinds of periodic arrays of holes, while Day et al. (1992) considered a material containing circular holes within a triangular or hexagonal matrix. Models with random circular holes, occasionally overlapping each other were also analyzed. Hu et al. (2000) developed a numerical model based on BEM to estimate the effective elastic properties, e.g. Young's modulus, bulk modulus and shear modulus. The main objective was to investigate the influence of the shortest distance between holes of random size and volume based on a normal distribution function. Cosmi (2004) introduced a new numerical model called the Cell Method (CM) to investigate the effect of randomly located voids. The model consisted of a homogeneous matrix of cells which contains randomly located voids. Li et al. (2009) developed an FEM model to calculate the elastic properties of porous materials with randomly distributed voids.

Although there are many theoretical and numerical approaches, the results are rather unsatisfactory because of the uncertainties in the characterization of the geometry changing from place to place horizontally and vertically. In this paper, we will use the random finite element method (RFEM) first developed by Griffiths and Fenton (1993) and Fenton and Griffiths (1993). In this method, conventional finite element analysis of a plane strain elastic solid using 8 nodes square elements (e.g. Smith and Griffiths 2004) is combined with random field generation (e.g. Fenton and Vanmarcke 1990, Fenton and Griffiths 2008) and Monte-Carlo simulations to develop output statistics of quantities such as the effective Young's modulus and Poisson's ratio. Not only the volume of inclusions can be considered by RFEM, but also the size of the inclusions through control of the spatial correlation length.

### 3 FINITE ELEMENT MODEL

The finite element mesh for this study considers a square plane strain block of material of unit side length modeled by  $50 \times 50$  8-node square elements of side length  $\Delta x = \Delta y = 0.02$  as shown in Figure 1. The boundary conditions allow vertical movement only of nodes on the left side, horizontal movement only of nodes on the bottom side, with the bottom-left corner node fixed. The vertical components of all nodal freedoms on the top loaded side were "tied", as were the horizontal components of all nodal freedoms on the right side. Tied freedoms are forced to move by the same amount in the analysis by giving them all the same freedom number during stiffness assembly. The tied freedom approach offers an elegant way of modeling a heterogeneous medium as an ideal element of material. Other methods employing stress or strain control may also be used to give similar outcomes.

The unit vertical force shown as  $Q$  in the figure is applied to the tied vertical freedom on the top of the square. The boundary conditions ensure that no matter what degree of heterogeneity is introduced, such as, for example, the darker regions in Figure 1 indicating voids, the mesh will always deform as an ideal element with the top surface remaining horizontal and the right side remaining vertical. From these vertical and horizontal movements under the application of a unit vertical pressure, the effective Young's modulus and Poisson's ratio can be easily back-figured from linear elastic theory.

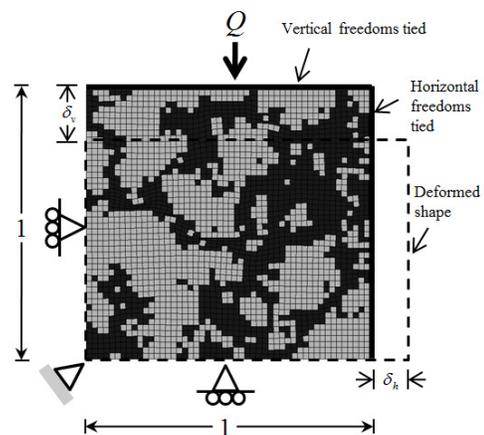


Figure 1. Tied freedom model with voids, under a unit vertical pressure.

### 4 RANDOM VOIDS MODEL

Input to the analysis consists of the target mean porosity  $n$  and spatial correlation length  $\theta$ , with the latter offering some degree of control over the void size. A random variable  $Z$  is assigned to each ele-

ment of the mesh based on a standard normal random field with spatial correlation length  $\theta$ . The random field generation properly accounts for local averaging, which is to say that the point variance of the random field is reduced as a function of the ratio  $\Delta x/\theta$ . In this study  $\Delta x/\theta \leq 10$ , so the variance reduction due to local averaging should be less than 10%. Once the standard normal random field values have been assigned to the mesh, cumulative distribution tables (suitably digitized in the software) are then used to estimate the value of the standard normal variable  $z_{n/2}$  for which

$$\Phi(z_{n/2}) - \Phi(0) = n/2 \quad (1)$$

as shown in Figure 2. Thereafter, any element assigned a random field value in the range  $|Z| > z_{n/2}$  is treated as intact material with Young's modulus and Poisson's ratio of  $E_0 = 1$  and  $\nu_0 = 0.3$ , respectively, while any element where  $|Z| \leq z_{n/2}$  is treated as a void element with an assigned Young's modulus of  $E = 0.01$  (100 times smaller than the surrounding intact material).

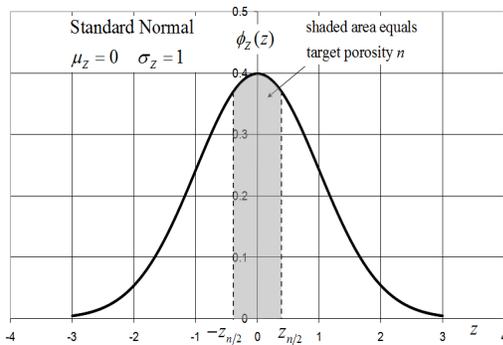


Figure 2. Target porosity area in standard normal random field.

## 5 VOID SIZE

It is clear that two rock or soil samples with the same porosity could have quite different void sizes. For example one material could include numerous small volume voids, while the other could include less frequent larger volume voids. The random field spatial correlation length  $\theta$  offers some quantitative control of void size. As in previous work by the authors, the Markov spatial correlation has been used as shown in Eq.2, where the correlation coefficient  $\rho$  between two points separated by an absolute distance  $|\tau|$  is given

$$\rho = \exp(-2|\tau|/\theta) \quad (2)$$

The function models an exponentially decaying correlation which ranges from 1 for points very close together to 0 for points a long way apart. Loosely speaking  $\theta$  is the distance within which

points are reasonably correlated (i.e.  $\rho > 0.14$ ). By changing the value of  $\theta$  in the parametric studies, the degree to which void elements with random values in the range  $|Z| \leq z_{n/2}$  tend to be clustered together can be influenced. A small value of  $\theta$  will imply fewer adjacent elements meeting the criterion at a given location, and hence smaller and more frequent voids while a large value of  $\theta$  will imply larger and less frequent voids as shown in Figure 3.

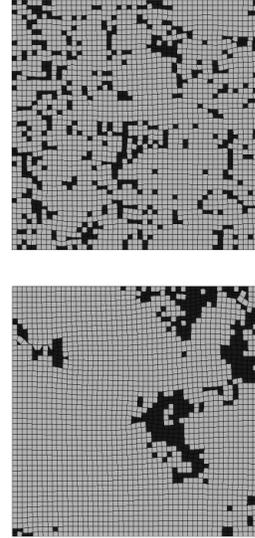


Figure 3. Typical simulations showing generation of voids at low and high spatial correlation lengths  $\theta$  ( $n=0.2$  in both cases).

## 6 MONTE-CARLO SIMULATIONS

A ‘‘Monte-Carlo’’ process means that analyses are repeated numerous times until the statistical properties of the output parameters start to stabilize. In this work, each Monte-Carlo simulation involves the generation of a random field and void distribution as explained previously. This is followed by an elastic analysis of the block such as that shown in Figure 1. The primary outputs from each elastic analysis are the vertical and horizontal deformations of the block,  $\delta_v$  and  $\delta_h$ , respectively. Although each simulation is based on the same  $\theta$  and  $n$ , the spatial location of the voids will be different each time, thus in some cases the voids may be just below the top of the block leading to a relatively high  $\delta_v$  while in others, the voids may be buried in the middle of the block leading to a relatively low  $\delta_v$ .

Following each simulation, the computed displacements  $\delta_v$  and  $\delta_h$  are converted into ‘‘effective’’ values of Young's modulus and Poisson's ratio from the plane strain elastic stress-strain relations to give

$$\nu = \frac{\delta_h}{\delta_h + \delta_v} \quad (3)$$

$$E = \frac{1 - \nu^2}{\delta_\nu} \quad (4)$$

Each Young's modulus value is then normalized as  $E/E_0$  by dividing by the intact Young's modulus  $E_0$ . Discussion of the Poisson's ratio response is left for future work.

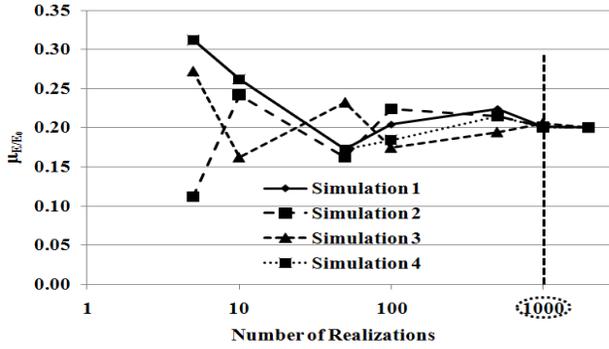


Figure 4. The accuracy of the estimated finite element statistics depend on the number of realizations for  $n = 0.3$  and  $\theta = 0.1$ .

## 7 RESULTS OF RFEM

In this study, 1000 simulations was shown to be enough to give statistically repeatable results as shown in Figure 4, so following each set of Monte-Carlo simulations, the mean and standard deviation of the normalized Young's modulus was computed for a range of parametric variations of  $n$  and  $\Theta$  with results shown in Figures 5 and 6 respectively. In the current work we have expressed the spatial correlation length in dimensionless form

$$\Theta = \frac{\theta}{L} \quad (5)$$

where  $L$  is the width of the loaded element in Figure 1 ( $L=1$ )

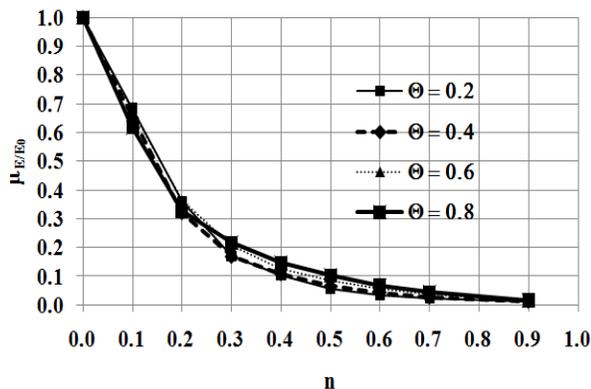


Figure 5.  $\mu_{E/E_0}$  vs.  $n$  for  $0.2 \leq \Theta \leq 0.8$ .

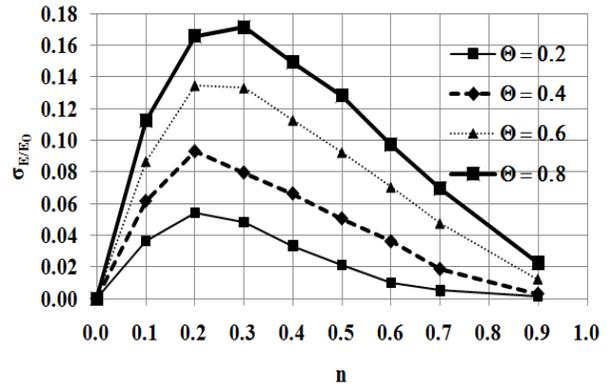


Figure 6.  $\sigma_{E/E_0}$  vs.  $n$  for  $0.2 \leq \Theta \leq 0.8$ .

It can be noted from Figure 5 that the mean Young's modulus  $\mu_E$  obviously drops towards zero with increasing porosity  $n$  and that  $\Theta$  does not have much influence on  $\mu_E$ . Figure 6 shows that  $\Theta$  has more influence on the standard deviation of Young's modulus  $\sigma_E$  and also displays a maximum at  $n \approx 0.3$ .

## 8 COMPARISON WITH OTHER SOLUTIONS

The current results are now compared with existing theoretical and numerical approaches. From Figure 7, it can be observed that the current method tends to give lower values of the effective Young's modulus than the theoretical methods described previously. Figure 8 shows a comparison between the current method and numerical results of Isida and Igawa (1991) and Day et al. (1992). Once more, the current method tends to give lower values of the effective Young's modulus, especially for lower values of  $n$ .

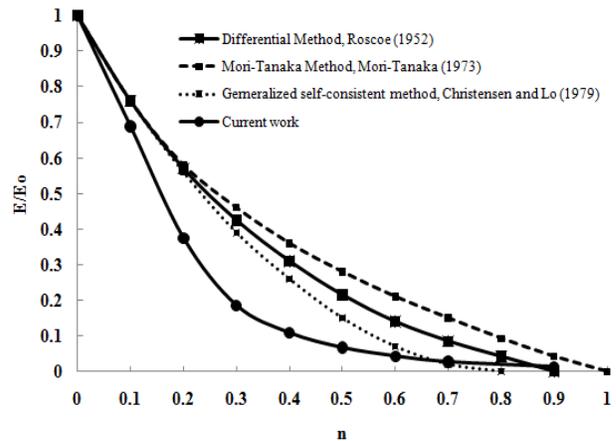


Figure 7. Comparison of the effective Young's modulus obtained from RFEM and different theoretical models.

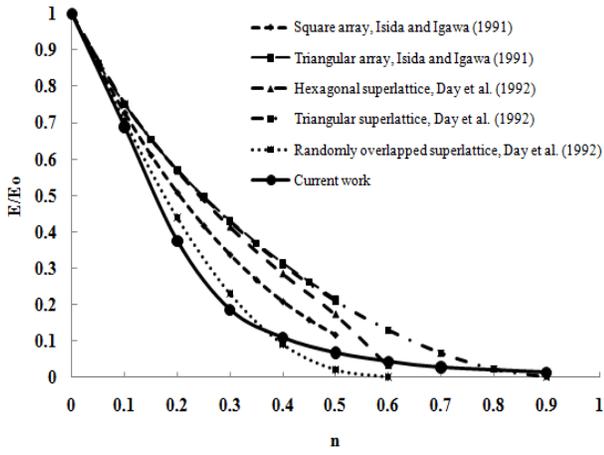


Figure 8. Comparison of the effective Young's modulus obtained from RFEM and different numerical models.

The lower values generated by the RFEM results suggests greater conservatism in design, but the reasons for this trend are under continued investigation.

## 9 FOUNDATION SETTLEMENT

A similar technique using RFEM has also been applied to study the influence of voids on foundation settlement using the mesh shown in Figure 9. The foundation sub-soil was modelled using  $50 \times 30$  square 8-node finite elements of unit side length. The strip footing at the ground surface was represented using tied vertical and horizontal freedoms covering ten elements with a width of 10 m. For a uniform foundation with no voids, a comparison was made with results of Sudret and Der Kiureghian (2000) and the computed vertical displacement of 0.043 m was in very good agreement. It may be noted that the vertical displacement of a rigid footing is approximately the same as the average settlement of the centre and edge of a flexible footing carrying the same total load (Davis and Taylor 1962, Poulos and Davis 1974). From the homogeneous validation example, the constant of proportionality relating the inverse of the effective Young's modulus to vertical footing displacement given by

$$\delta_v = \frac{C}{E} \quad (6)$$

was found to be  $C = 2.162 \text{ m}^3/\text{MN}$ .

This constant was then used to compute the effective Young's modulus during 1000 Monte-Carlo simulations based on the vertical footing displacement, using

$$E_i = \frac{C}{\delta_{v_i}}, \quad i = 1, 2, \dots, 1000 \quad (7)$$

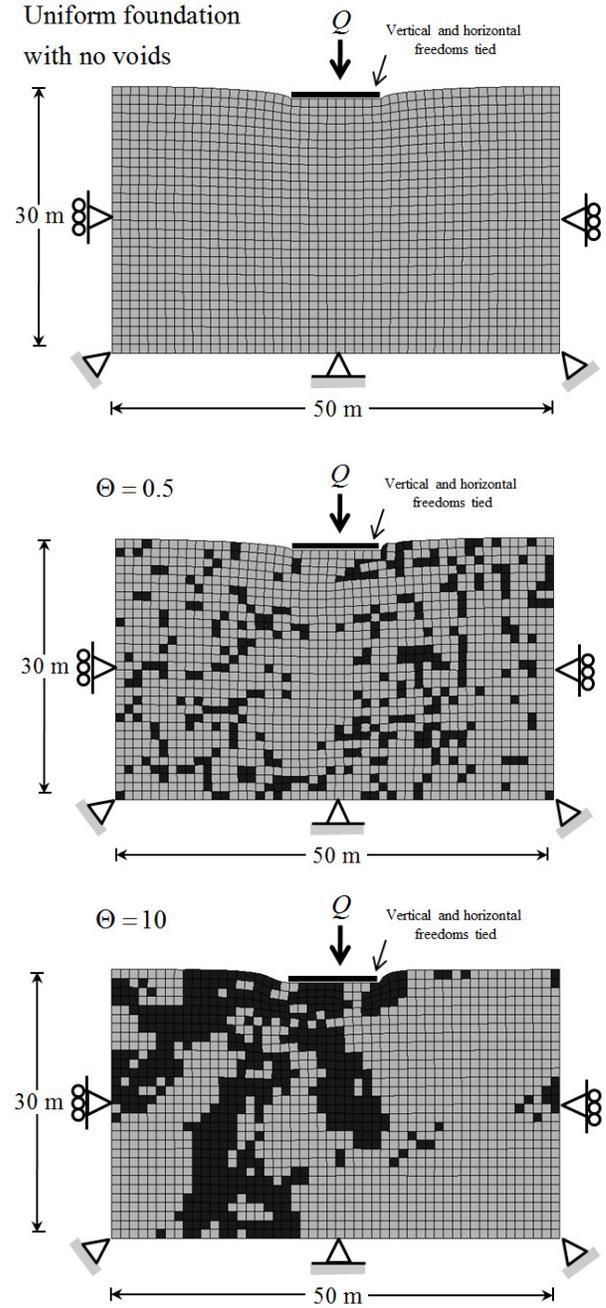


Figure 9. Tied freedom analysis in foundation settlement on material with no voids and material containing voids with  $n = 0.2$  and  $\Theta = 0.5$  and  $10.0$  ( $Q = 0.2 \text{ MPa}$ ,  $E_0 = 50 \text{ MPa}$ ,  $\nu = 0.3$  and  $B = 10 \text{ m}$ ).

The results of Monte-Carlo simulations for a range of parametric variations of  $n$  and  $\Theta$  are presented in Figure 10. In the settlement analyses, we have expressed the spatial correlation length  $\Theta$  in dimensionless form by dividing the actual correlation length  $\theta$  by the footing width  $B = 10 \text{ m}$ . Each computed value of  $E$  from Eq. (4) is also expressed in dimensionless form by dividing by the intact value  $E_0 = 50 \text{ MPa}$ . Figure 10 shows reasonable agreement between the effective Young's modulus from the settlement analysis and element tests for a range of porosities. Further work is needed in this area to assess the scaling effects.

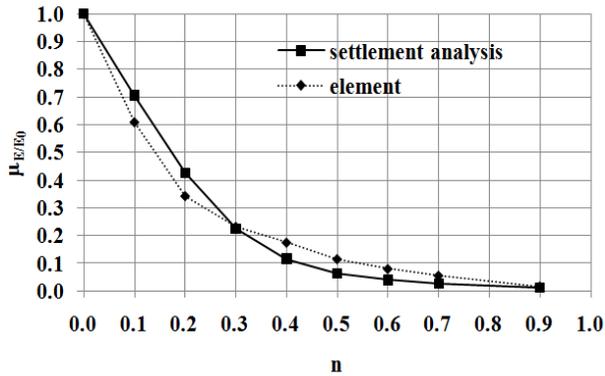


Figure 10. Comparison of results from settlement analysis and element test with  $\mu_{E/E_0}$  vs.  $n$  for  $\Theta = 1.0$ .

## 10 PROBABILISTIC INTERPRETATION

In order to make probabilistic interpretations from a Monte-Carlo analysis, we can either count the number of simulations that exceed an allowable design value as a proportion of the total number of simulations, or we can fit a probability density function to the data as in Figure 11. The histogram shown in the figure indicates the frequency distribution of effective Young's modulus values following a suite of 1000 Monte-Carlo simulations. The smooth line is a fitted normal distribution based on the computed mean and standard deviation values ( $\mu_{E/E_0} = 0.454$ ,  $\sigma_{E/E_0} = 0.074$ ).

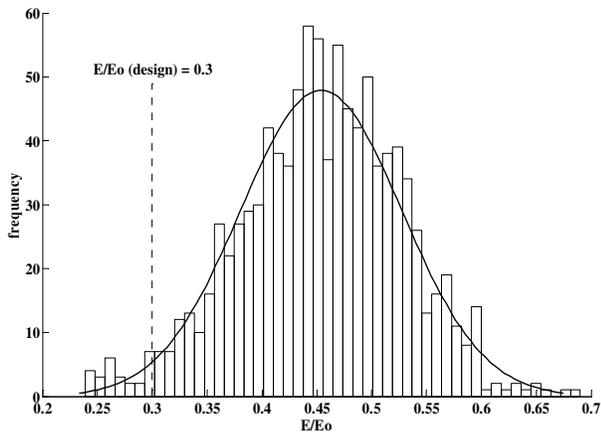


Figure 11. Histogram of Young's modulus values following a suite of Monte-Carlo analyses together with a fitted normal distribution.  $n = 0.2$ ,  $\Theta = 0.5$ .

Let us assume that the design of a footing has failed if  $E/E_0 < 0.3$ . Thus for the particular parametric combination of  $n = 0.2$  and  $\Theta = 0.5$  we might wish to estimate  $P[E/E_0 < 0.3]$ .

Sample calculation:

- 1) For input values  $n = 0.2$  and  $\Theta = 0.5$
- 2) From Monte-Carlo simulations,  
 $\mu_{E/E_0} = 0.454$ ,  $\sigma_{E/E_0} = 0.074$
- 3) Probability of design "failure"

$$P[E/E_0 < 0.3] = \Phi\left[\frac{0.3 - 0.454}{0.074}\right] = 1.88\%$$

where  $\Phi[\cdot]$  is the standard cumulative distribution function.

## 11 CONCLUSION

RFEM shows promise as a powerful alternative approach for modeling the mechanical influence of inclusions and voids in geomaterials. The voids are not restricted to being simple shapes as in some of the theoretical methods, and the user can control the void volume and size through spatial correlation, thus two rocks with the same porosity could have quite different void sizes.

The RFEM has been used in this study to investigate the influence of porosity and void size on the effective stiffness of geomaterials containing random voids. A novel "tied freedom" approach has been used to model an idealized element test leading to predictions of the effective Young's modulus as a function of porosity and void size. Some preliminary studies of foundation settlement on sub-soils containing voids were also presented using a similar RFEM methodology and it was demonstrated how these results could be used to deliver probabilistic conclusion regarding foundation settlement.

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